

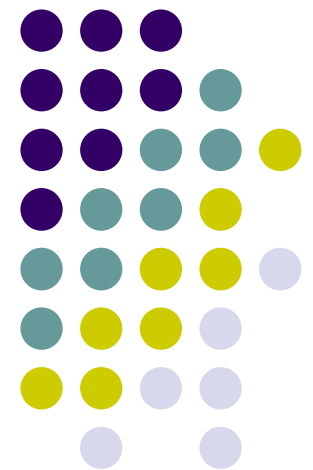
Computer Graphics

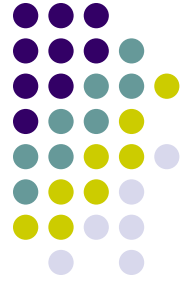
CS 543 – Lecture 3 (Part 2)

Linear Algebra for Graphics (Points, Scalars, Vectors)

Prof Emmanuel Agu

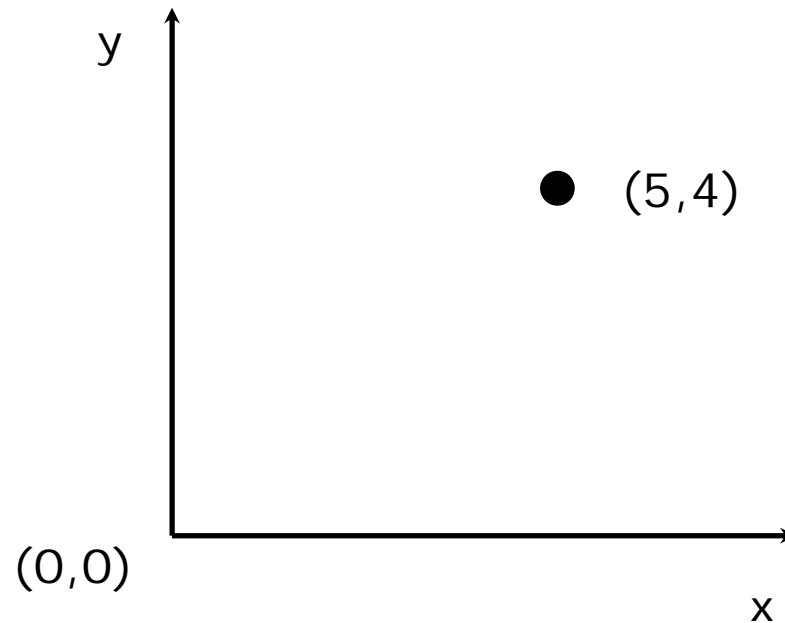
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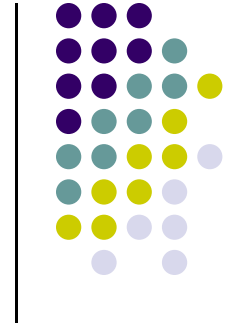


Points, Scalars and Vectors

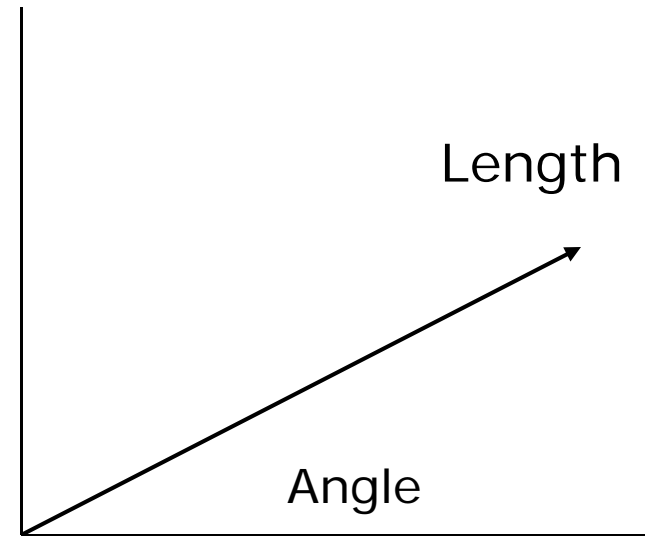
- Points, vectors defined relative to a coordinate system
- Example: Point $(5,4)$



Vectors

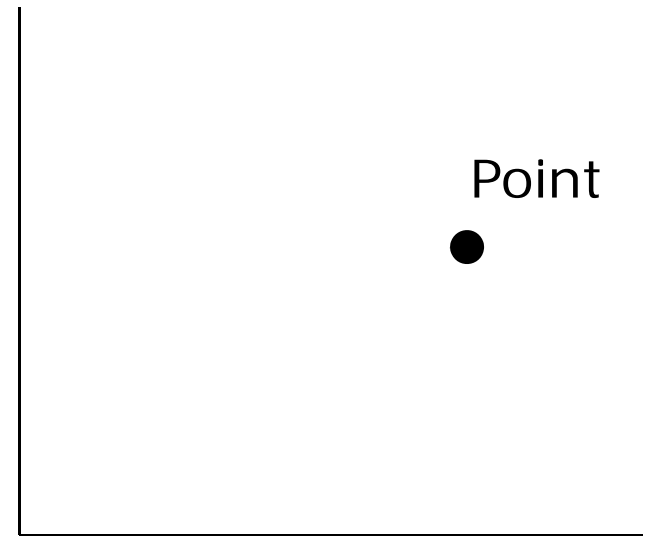


- Magnitude
- Direction
- **NO** position
- Can be added, scaled, rotated
- CG vectors: 2, 3 or 4 dimensions

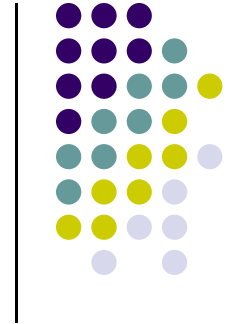


Points

- Location in coordinate system
- Cannot add or scale
- Subtract 2 points = vector



Vector-Point Relationship

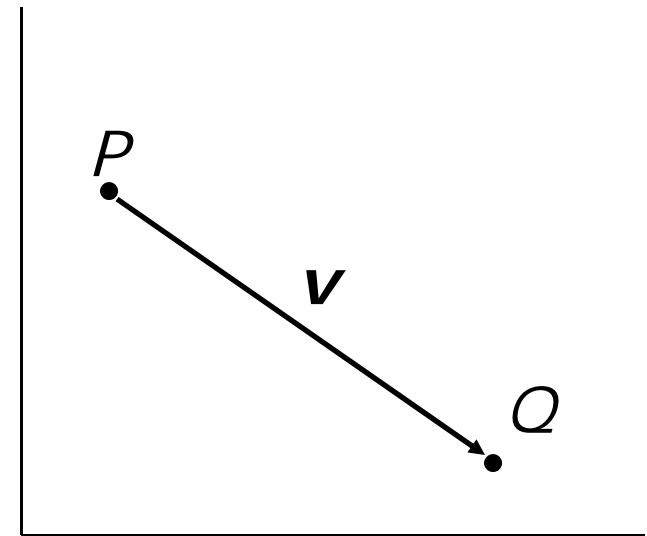


- Diff. b/w 2 points = vector

$$\mathbf{v} = Q - P$$

- point + vector = point

$$\mathbf{v} + P = Q$$



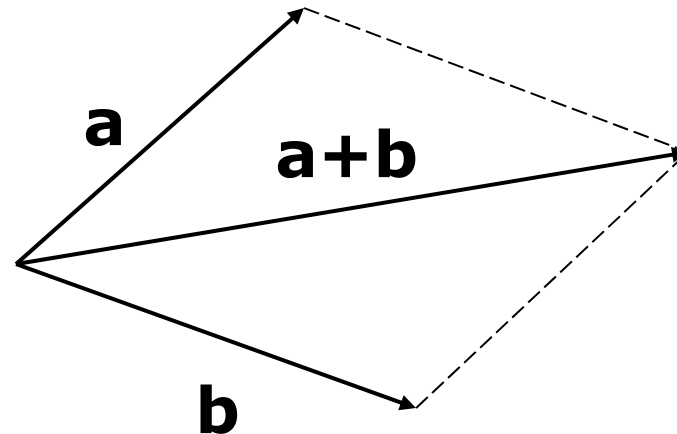
Vector Operations



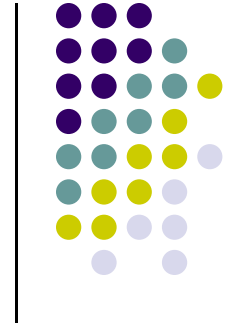
- Define vectors
 $\mathbf{a} = (a_1, a_2, a_3)$
 $\mathbf{b} = (b_1, b_2, b_3)$

Then vector addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

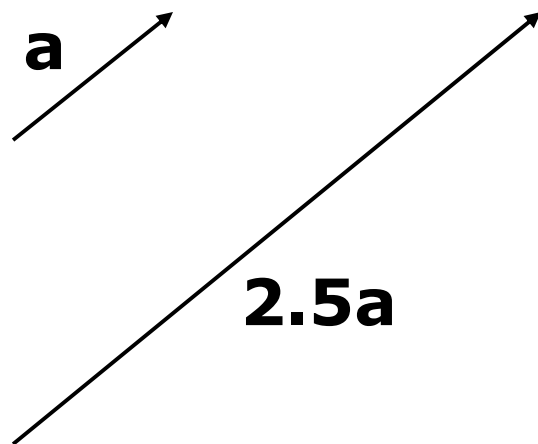


Vector Operations



- Define scalar, s
- Scaling vector by a scalar

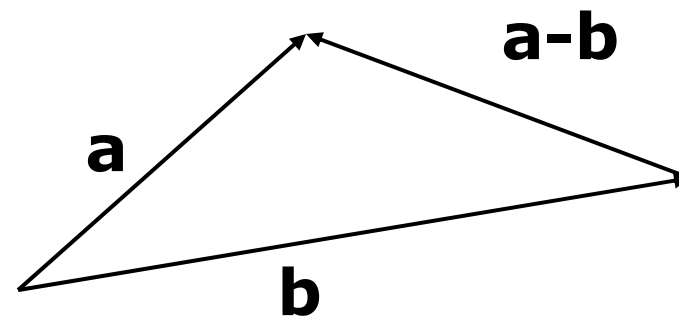
$$\mathbf{as} = (a_1s, a_2s, a_3s)$$

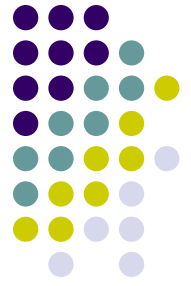


Note vector subtraction:

$$\mathbf{a} - \mathbf{b}$$

$$= (a_1 + (-b_1), a_2 + (-b_2), a_3 + (-b_3))$$





Vector Operations: Examples

- Scaling vector by a scalar

$$\mathbf{a}s = (a_1s, a_2s, a_3s)$$

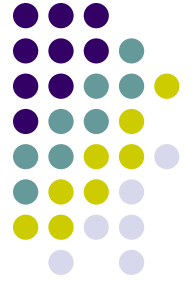
- Vector addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

- For example, if $\mathbf{a}=(2,5,6)$ and $\mathbf{b}=(-2,7,1)$ and $s=6$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = (0, 12, 7)$$

$$\mathbf{a}s = (a_1s, a_2s, a_3s) = (12, 30, 36)$$



Affine Combination

- Given a vector

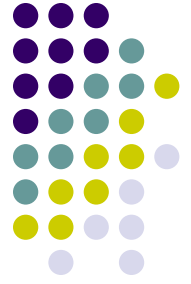
$$\mathbf{a} = (a_1, a_2, a_3, \dots, a_n)$$

$$a_1 + a_2 + \dots + a_n = 1$$

- Affine combination: Sum of all components = 1

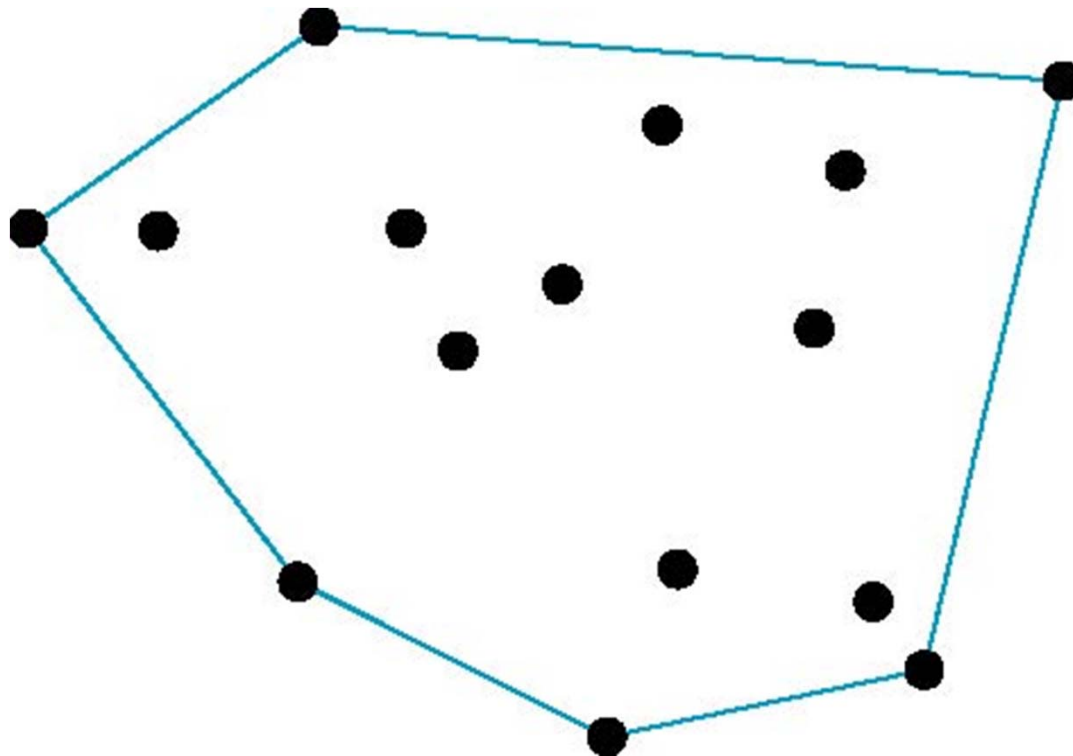
$$a_1, a_2, \dots, a_n = \textit{non-negative}$$

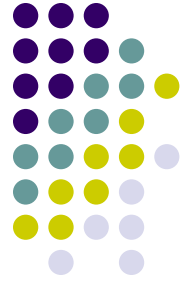
- Convex affine = affine + no negative component



Convex Hull

- Smallest convex object containing P_1, P_2, \dots, P_n
- Formed by “shrink wrapping” points





Magnitude of a Vector

- Magnitude of \mathbf{a}

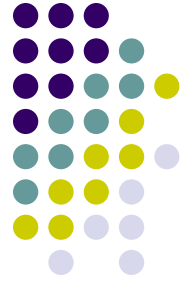
$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

- Normalizing a vector (unit vector)

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{vector}}{\text{magnitude}}$$

- Note magnitude of normalized vector = 1. i.e

$$\sqrt{a_1^2 + a_2^2 + \dots + a_n^2} = 1$$



Dot Product (Scalar product)

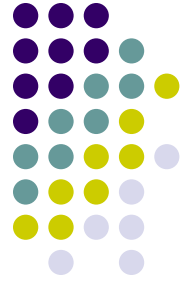
- Dot product,

$$d = \mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n$$

- For example, if $a=(2,3,1)$ and $b=(0,4,-1)$

then

$$\begin{aligned} a \cdot b &= (2 \times 0) + (3 \times 4) + (1 \times -1) \\ &= 0 + 12 - 1 = 11 \end{aligned}$$



Properties of Dot Products

- Symmetry (or commutative):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- Linearity:

$$(\mathbf{a} + \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$$

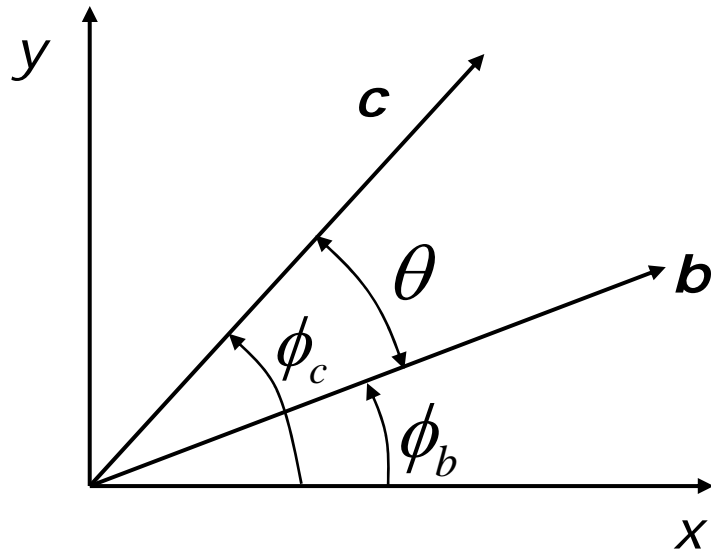
- Homogeneity:

$$(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$$

- And

$$|\mathbf{b}^2| = \mathbf{b} \cdot \mathbf{b}$$

Angle Between Two Vectors

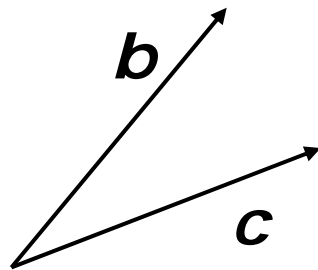


$$\mathbf{b} = (|\mathbf{b}| \cos \phi_b, |\mathbf{b}| \sin \phi_b)$$

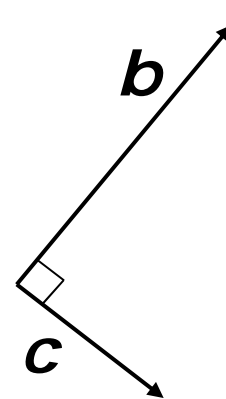
$$\mathbf{c} = (|\mathbf{c}| \cos \phi_c, |\mathbf{c}| \sin \phi_c)$$

$$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}| |\mathbf{c}| \cos \theta$$

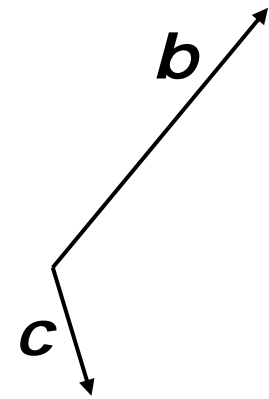
Sign of $\mathbf{b} \cdot \mathbf{c}$:



$$\mathbf{b} \cdot \mathbf{c} > 0$$

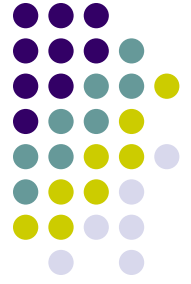


$$\mathbf{b} \cdot \mathbf{c} = 0$$

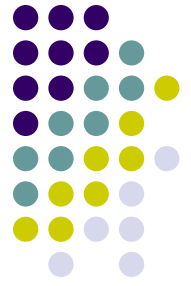


$$\mathbf{b} \cdot \mathbf{c} < 0$$

Angle Between Two Vectors



- Find the angle b/w the vectors $\mathbf{b} = (3,4)$ and $\mathbf{c} = (5,2)$



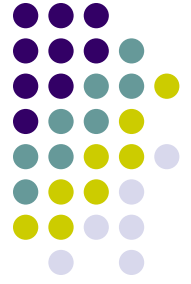
Angle Between Two Vectors

- Find the angle b/w the vectors $\mathbf{b} = (3,4)$ and $\mathbf{c} = (5,2)$
 - $|\mathbf{b}| = 5$, $|\mathbf{c}| = 5.385$

$$\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5} \right) \quad \hat{\mathbf{b}} \cdot \hat{\mathbf{c}} = 0.85422 = \cos \theta$$

$$\theta = 31.326^\circ$$

Standard Unit Vectors

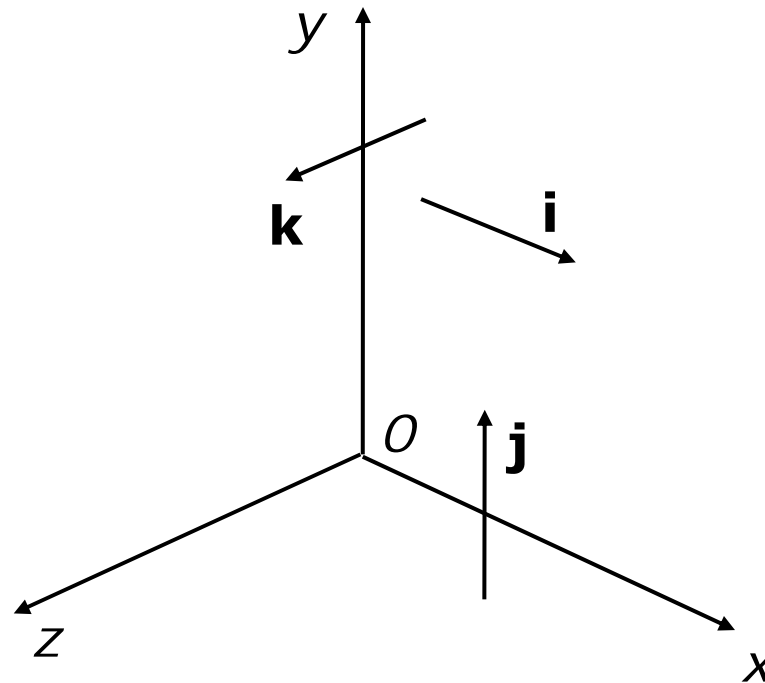


Define

$$\mathbf{i} = (1,0,0)$$

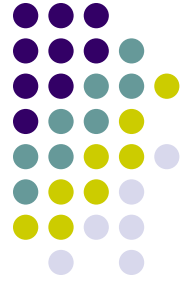
$$\mathbf{j} = (0,1,0)$$

$$\mathbf{k} = (0,0,1)$$



So that any vector,

$$\mathbf{v} = (a,b,c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$



Cross Product (Vector product)

If

$$\mathbf{a} = (a_x, a_y, a_z) \quad \mathbf{b} = (b_x, b_y, b_z)$$

Then

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}$$

Remember using determinant

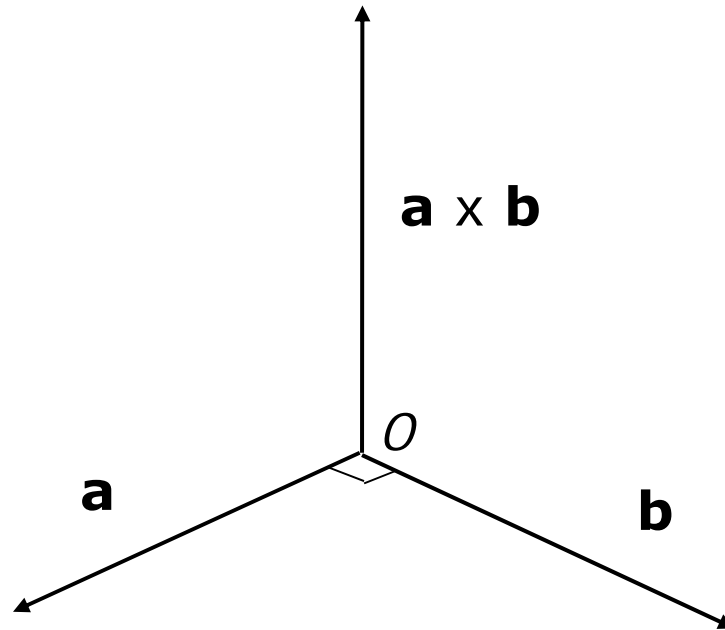
$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Note: $\mathbf{a} \times \mathbf{b}$ is perpendicular to \mathbf{a} and \mathbf{b}

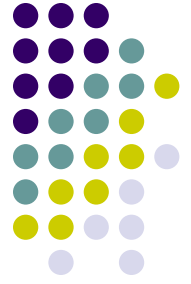
Cross Product



Note: $\mathbf{a} \times \mathbf{b}$ is perpendicular to both \mathbf{a} and \mathbf{b}

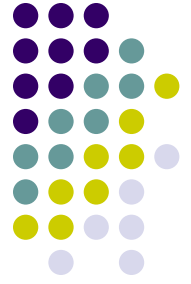


Cross Product



Calculate $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = (3, 0, 2)$ and $\mathbf{b} = (4, 1, 8)$

Cross Product



Calculate $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = (3, 0, 2)$ and $\mathbf{b} = (4, 1, 8)$

$$\mathbf{a} \times \mathbf{b} = -2\mathbf{i} - 16\mathbf{j} + 3\mathbf{k}$$

Finding Vector Reflected From a Surface



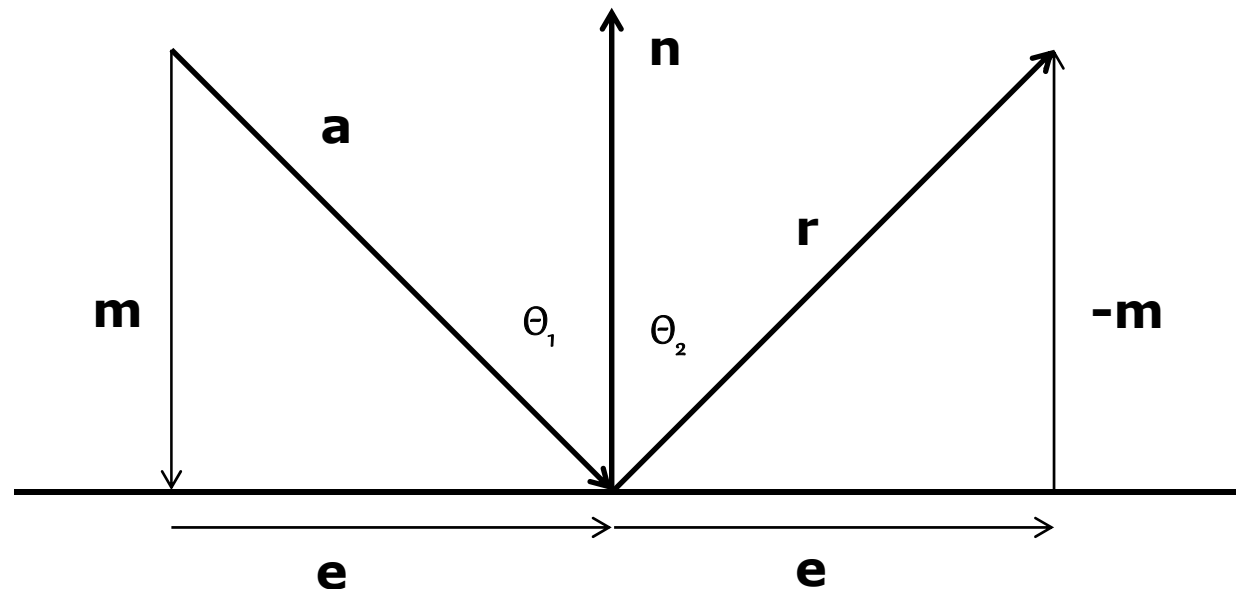
- \mathbf{a} = original vector
- \mathbf{n} = normal vector
- \mathbf{r} = reflected vector
- \mathbf{m} = projection of \mathbf{a} along \mathbf{n}
- \mathbf{e} = projection of \mathbf{a} orthogonal to \mathbf{n}

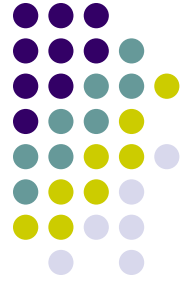
Note: $\theta_1 = \theta_2$

$$\mathbf{r} = \mathbf{e} - \mathbf{m}$$

$$\mathbf{e} = \mathbf{a} - \mathbf{m}$$

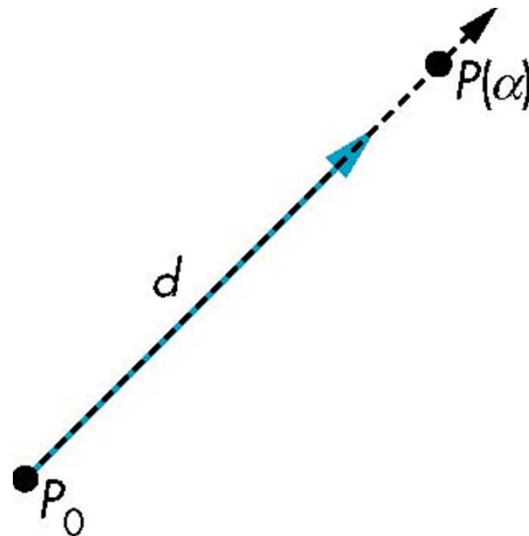
$$\Rightarrow \mathbf{r} = \mathbf{a} - 2\mathbf{m}$$





Lines

- Consider all points of the form
 - $P(\alpha) = P_0 + \alpha \mathbf{d}$
 - Set of all points that pass through P_0 in the direction of the vector \mathbf{d}





Parametric Form

- Two-dimensional forms of a line

- Explicit: $y = mx + h$
- Implicit: $ax + by + c = 0$
- Parametric:

$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$

$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

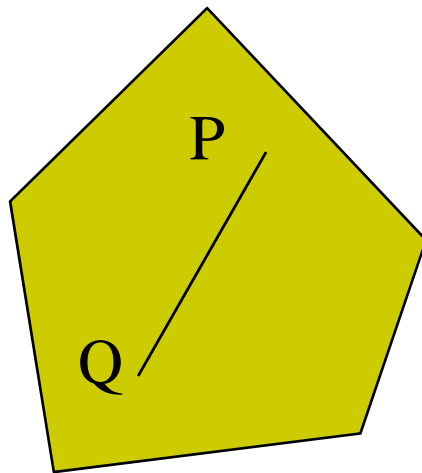
- Parametric form of line

- More robust and general than other forms
- Extends to curves and surfaces

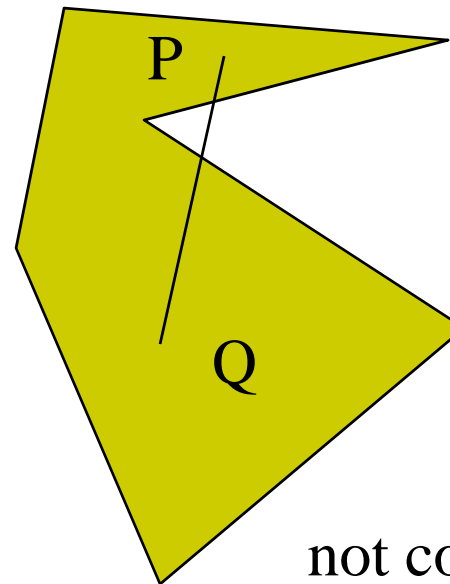


Convexity

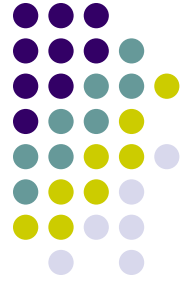
- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



convex



not convex

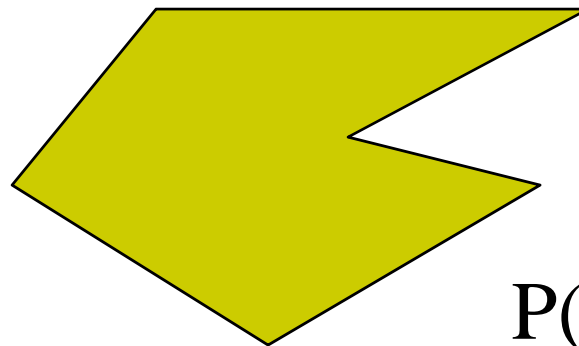


Curves and Surfaces

- Curves are one parameter entities of the form $P(\alpha)$ where the function is nonlinear
- Surfaces are formed from two-parameter functions $P(\alpha, \beta)$
 - Linear functions give planes and polygons



$P(\alpha)$



$P(\alpha, \beta)$

References

- Angel and Shreiner
- Hill, chapter 4.2 - 4.4

