Computer Graphics
CS 543 Lecture 13 (Part 2)
Curves

Prof Emmanuel Agu

Computer Science Dept.
Worcester Polytechnic Institute (WPI)

So Far…

- Dealt with straight lines and flat surfaces
- Real world objects include curves
- Need to develop:
    - Representations of curves
    - Tools to render curves

Curve Representation: Explicit

- One variable expressed in terms of another
- Example:

$$z = f(x, y)$$

- Works if one x-value for each y value
- Example: does not work for a sphere

$$z = \sqrt{x^2 + y^2}$$

- Rarely used in CG because of this limitation

Curve Representation: Implicit

- Represent 2D curve or 3D surface as zeros of a formula
- Example: sphere representation

$$x^2 + y^2 + z^2 - 1 = 0$$

- May limit classes of functions used
- Polynomial: function which can be expressed as linear combination of integer powers of x, y, z
- Degree of algebraic function: highest power in function
- Example: mx^4 has degree of 4

Curve Representation: Parametric

- Represent 2D curve as 2 functions, 1 parameter

$$(x(u), y(u))$$

- 3D surface as 3 functions, 2 parameters

$$(x(u,v), y(u,v), z(u,v))$$

- Example: parametric sphere

$$x(\theta,\phi) = \cos\phi\cos\theta$$

$$y(\theta,\phi) = \cos\phi\sin\theta$$

$$z(\theta,\phi) = \sin\phi$$

Choosing Representations

- Different representation suitable for different applications
- Implicit representations good for:
  - Computing ray intersection with surface
  - Determing if point is inside/outside a surface
- Parametric representation good for:
  - Breaking surface into small polygonal elements for rendering
  - Subdivide into smaller patches
- Sometimes possible to convert one representation into another

Continuity

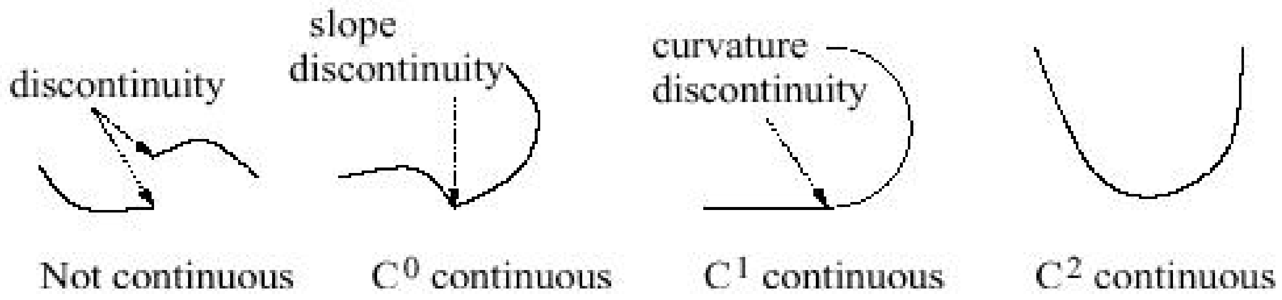
- Consider parametric curve

$$P(u) = (x(u), y(u), z(u))^T$$

- We would like smoothest curves possible
- Mathematically express smoothness as continuity (no jumps)
- **Defn:** if kth derivatives exist, and are continuous, curve has kth order parametric continuity denoted C^k

Continuity

- 0th order means curve is continuous
- 1st order means curve tangent vectors vary continuously, etc

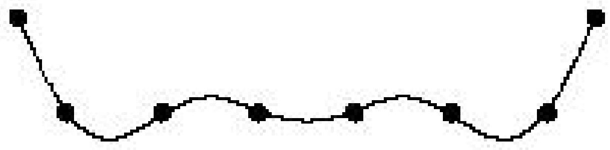


Interactive Curve Design

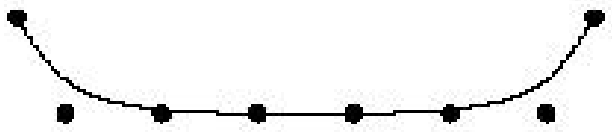
- Mathematical formula unsuitable for designers
- Prefer to interactively give sequence of points (control points)
- Write procedure:
  - **Input:** sequence of points
  - **Output:** parametric representation of curve

Interactive Curve Design

- 1 approach: curves pass through control points (interpolate)
- **Example:** Lagrangian Interpolating Polynomial
- Difficulty with this approach:
  - Polynomials always have “wiggles”
  - For straight lines wiggling is a problem
- Our approach: approximate control points (Bezier, B-Splines)



Interpolation

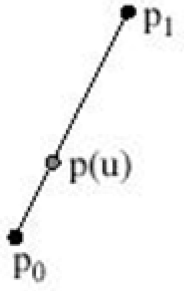


Approximation

De Casteljau Algorithm

- Consider smooth curve that approximates sequence of control points [p0,p1,….]

$$p(u) = (1-u)p_0 + up_1 \qquad 0 \le u \le 1$$

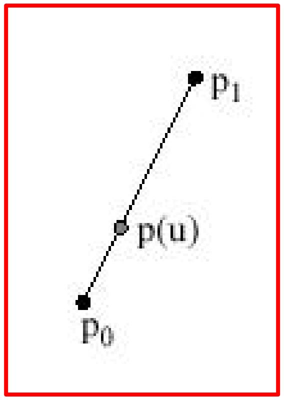


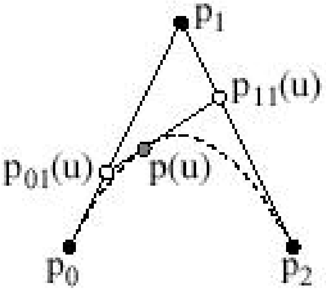
- Blending functions: *u* and *(1 – u)* are non-negative and sum to one

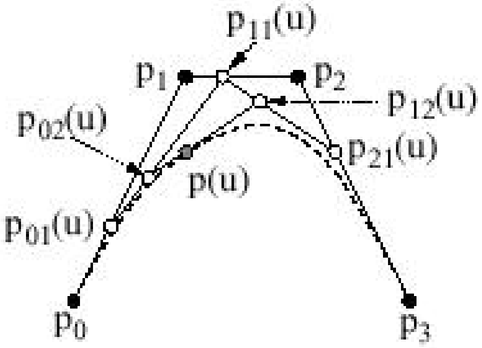
De Casteljau Algorithm

- Now consider 3 points
- 2 line segments, P0 to P1 and P1 to P2

$$p_{01}(u) = (1-u)\,p_0 + u p_1 \qquad p_{11}(u) = (1-u)\,p_1 + u p_2$$







De Casteljau Algorithm

Substituting known values of $p_{01}(u)$ and $p_{11}(u)$

$$p(u) = (1-u)\,p_{01} + u p_{11}(u)$$

$$= \underbrace{(1-u)^2}_{b_{02}(u)} \boxed{p_0} + \underbrace{(2u(1-u))}_{b_{12}(u)} \boxed{p_1} + \underbrace{u^2}_{b_{22}(u)} \boxed{p_2}$$

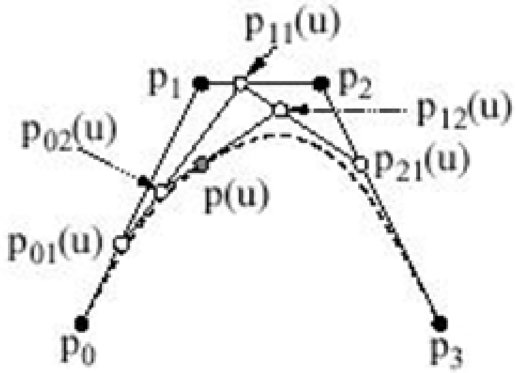
Blending functions for degree 2 Bezier curve

$$b_{02}(u) = (1-u)^2 \qquad b_{12}(u) = 2u(1-u) \qquad b_{22}(u) = u^2$$

Note: blending functions, non-negative, sum to 1

De Casteljau Algorithm

- Extend to 4 control points P0, P1, P2, P3



$$p(u)=\underbrace{(1-u)^3}_{b_{03}(u)}\boxed{p_0}+\underbrace{(3u(1-u)^2)}_{b_{13}(u)}\boxed{p_1}+\underbrace{(3u^2(1-u))}_{b_{23}(u)}\boxed{p_2}+\underbrace{u^3}_{b_{33}(u)}$$

- Final result above is Bezier curve of degree 3

De Casteljau Algorithm

$$p(u) = \underbrace{(1-u)^3}_{b_{03}(u)}\boxed{p_0} + \underbrace{(3u(1-u)^2)}_{b_{13}(u)}\boxed{p_1} + \underbrace{(3u^2(1-u))}_{b_{23}(u)}\boxed{p_2} + \underbrace{u^3}_{b_{33}(u)}$$

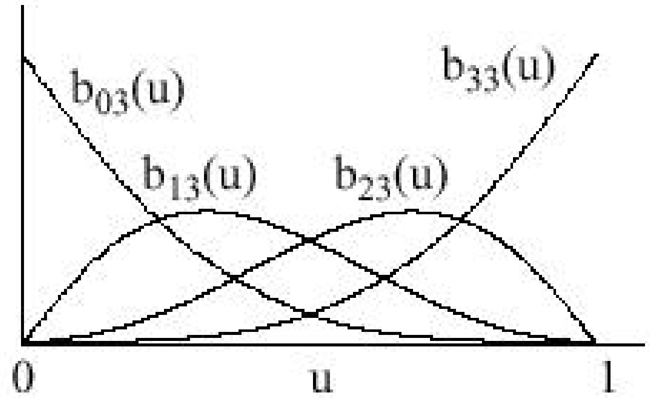
- Blending functions are polynomial functions called **Bernstein's polynomials**

$$b_{03}(u) = (1-u)^3$$

$$b_{13}(u) = 3u(1-u)^2$$

$$b_{23}(u) = 3u^2(1-u)$$

$$b_{33}(u) = u^3$$



De Casteljau Algorithm

$$p(u) = \underset{\underset{1}{\uparrow}}{(1-u)^3}\,\boxed{p_0} + \underset{\underset{3}{\uparrow}}{(3u(1-u)^2)}\,\boxed{p_1} + \underset{\underset{3}{\uparrow}}{(3u^2(1-u))}\,\boxed{p_2} + \underset{\underset{1}{\uparrow}}{u^3}$$

- Writing coefficient of blending functions gives Pascal’s triangle

1

1 1

1 2 1 ← 3 control points

1 3 3 1 ← 4 control points

1 4 6 4 1 ← 5 control points

De Casteljau Algorithm

- In general, blending function for k Bezier curve has form

$$b_{ik}(u) = \binom{k}{i}(1-u)^{k-i}u^{i}$$

- Example

$$b_{03}(u) = \binom{3}{0}(1-u)^{3-0}u^{0} = (1-u)^{3}$$

De Casteljau Algorithm

- Can express cubic parametric curve in matrix form

$$p(u) = [1, u, u^2, u^3] M_B \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

where

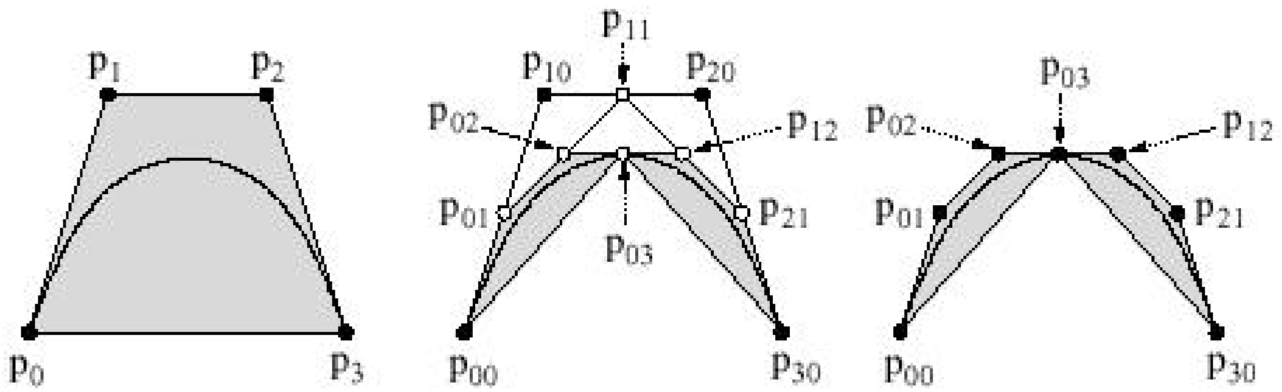
$$M_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Subdividing Bezier Curves

- OpenGL renders flat objects
- To render curves, approximate with small linear segments
- Subdivide surface to polygonal patches
- Bezier curves useful for elegant, recursive subdivision

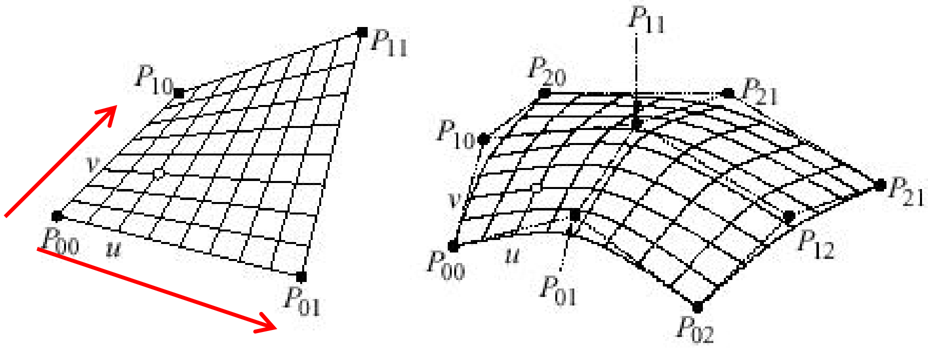
Subdividing Bezier Curves

- Let (P0… P3) denote original sequence of control points
- Recursively interpolate with u = ½ as below
- Sequences (P00,P01,P02,P03) and (P03,P12,P21,30) define Bezier curves also
- Bezier Curves can either be straightened or curved recursively in this way



Bezier Surfaces

- Bezier surfaces: interpolate in two dimensions
- This called Bilinear interpolation
- Example: 4 control points, P00, P01, P10, P11, 2 parameters **u** and **v**
- Interpolate between
  - P00 and P01 using u
  - P10 and P11 using u
  - P00 and P10 using v
  - P01 and P11 using v



$$p(u,v) = (1-v)((1-u)\,p_{00} + up_{01}) + v((1-u)\,p_{10} + up_{11})$$

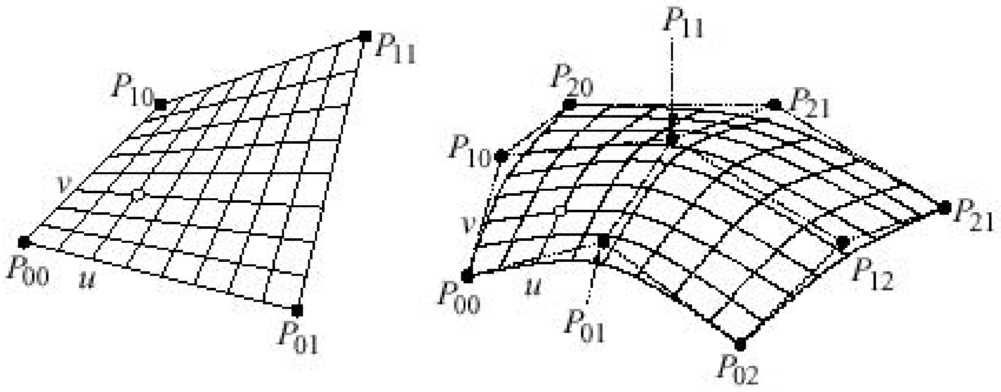
Bezier Surfaces

- Expressing in terms of blending functions

$$p(u,v) = b_{01}(v)b_{01}(u)\,p_{00} + b_{01}(v)b_{11}b_{01}(u)\,p_{01} + b_{11}(v)b_{11}(u)\,p_{11}$$

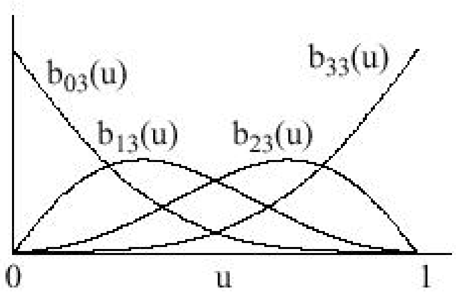
Generalizing

$$p(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(v)b_{j,3}(u)\,p_{i,j}$$



Problems with Bezier Curves

- Bezier curves are elegant but too many control points
- To achieve smoother curve
  - = more control points
  - = higher order polynomial
  - = more calculations

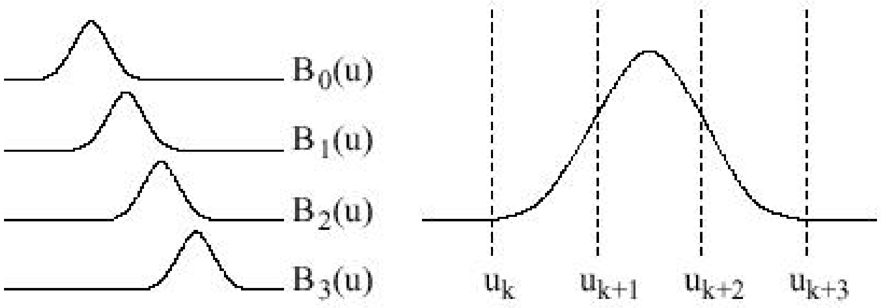


- **Global support problem:** All blending functions are non-zero for all values of *u*
- All control points contribute to all parts of the curve
- Means after modelling complex surface (e.g. a ship), if one control point is moves, recalculate everything!

B-Splines

- B-splines designed to address Bezier shortcomings
- B-Spline given by blending control points
- **Local support:** Each spline contributes in limited range
- Only non-zero splines contribute in a given range of *u*

$$p(u)=\sum_{i=0}^{m} B_i(u)\,p_i$$



B-spline blending functions, order 2

NURBS

- Encompasses both Bezier curves/surfaces and B-splines
- Non-uniform Rational B-splines (NURBS)
- Rational function is ratio of two polynomials
- Some curves can be expressed as rational functions but not as simple polynomials
- No known exact polynomial for circle
- Rational parametrization of unit circle on xy-plane:

$$x(u) = \frac{1-u^2}{1+u^2}$$

$$y(u) = \frac{2u}{1+u^2}$$

$$z(u) = 0$$

NURBS

- We can apply homogeneous coordinates to bring in *w*

$$x(u) = 1 - u^2$$

$$y(u) = 2u$$

$$z(u) = 0$$

$$w(u) = 1 + u^2$$

- Using w, we get we cleanly integrate rational parametrization
- Useful property of NURBS: preserved under transformation

Rendering Curves

- Tesselation shaders can now be written on the GPU to render curves

References

- Hill and Kelley, chapter 11
- Angel and Shreiner, Interactive Computer Graphics, 6th edition, Chapter 10