Computer Graphics 543
Lecture 4 (Part 1): Rotations and Matrix Concatenation

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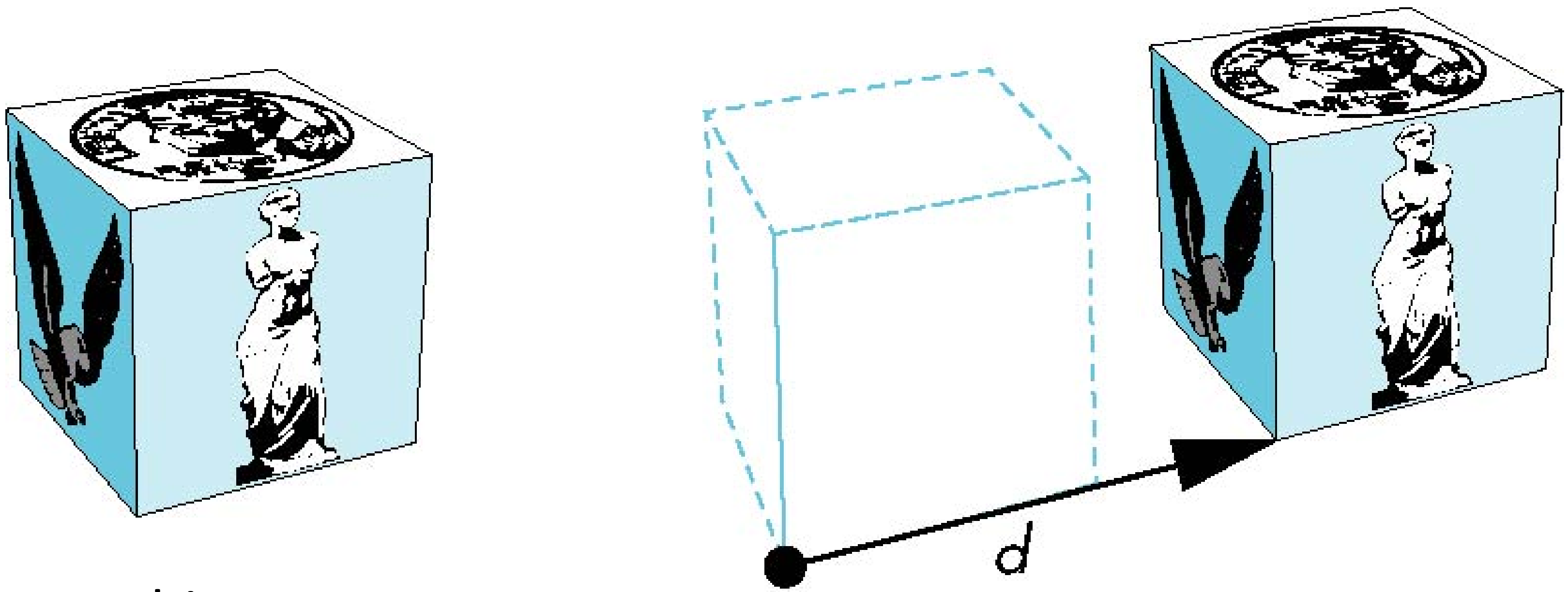
Recall: 3D Translation

- **Translate:** Move each vertex by same distance **d = (t_x, t_y, t_z)**



object

translation: every vertex displaced by same vector

Recall: 3D Translation Matrix

- In 3D :

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$

Translate(tx,ty,tz)

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- Where: *x'= x.1* + *y.0* + *z.0* + *tx.1* = *x* + *tx*, … etc

Recall: Scaling

Scale: Expand or contract along each axis (fixed point of origin)

$\mathbf{S} = \mathbf{S}(s_x, s_y, s_z)$

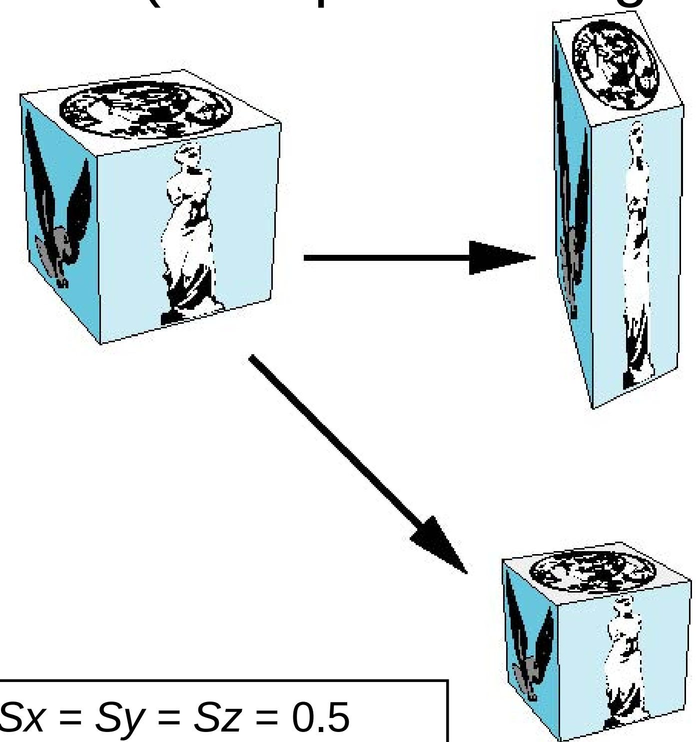
$x' = s_x x$

$y' = s_y x$

$z' = s_z x$

$\mathbf{p}' = \mathbf{S}\mathbf{p}$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

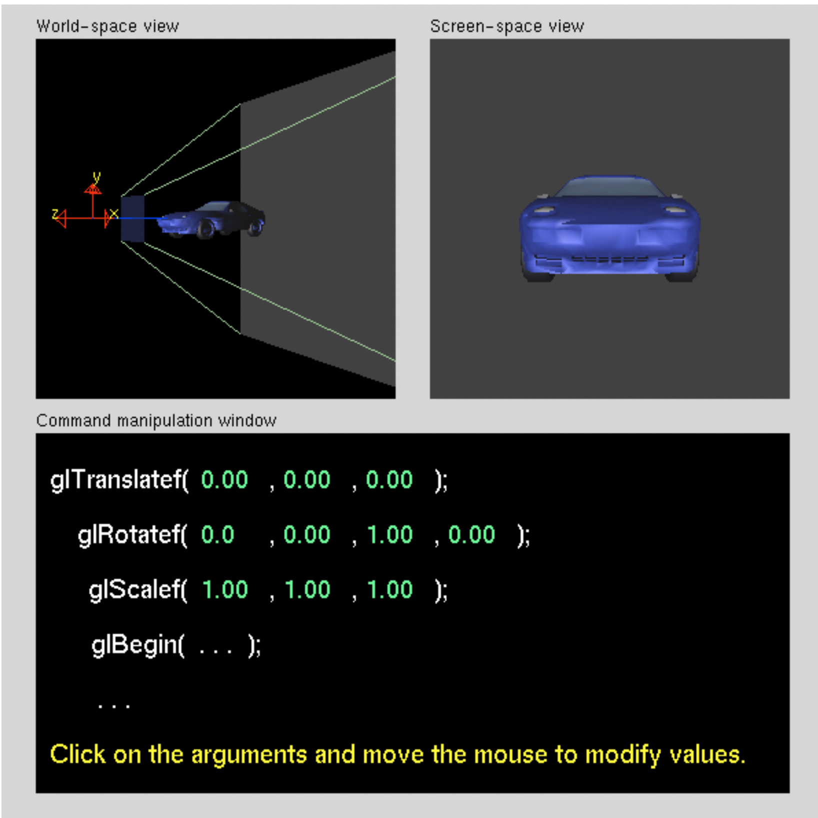


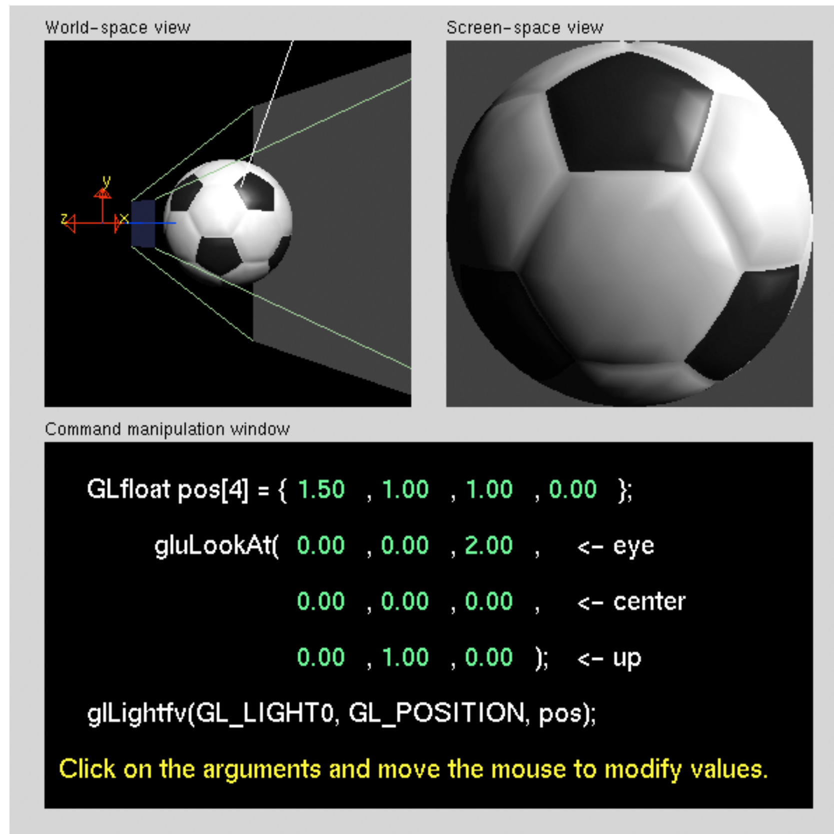
- **Example:** *Sx* = *Sy* = *Sz* = 0.5

scales big cube (sides = 1)

to small cube (sides = 0.5)

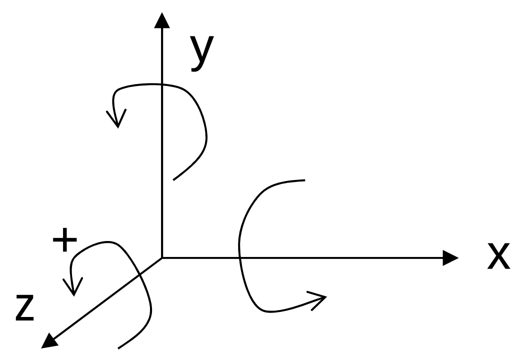
Nate Robbins Translate, Scale Rotate Demo





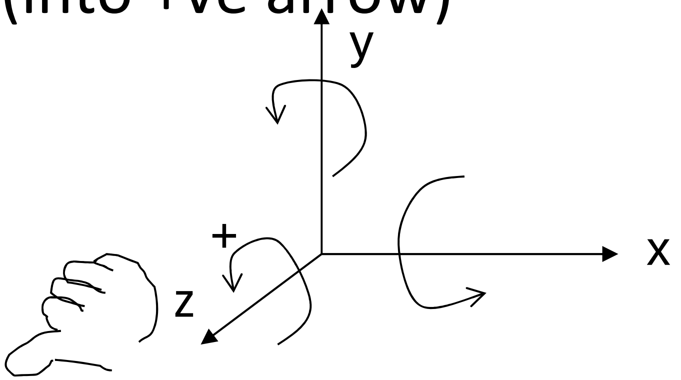
Rotating in 3D

- Many degrees of freedom. Rotate about what axis?
- 3D rotation: about a defined axis
- Different transform matrix for:
  - Rotation about x-axis
  - Rotation about y-axis
  - Rotation about z-axis

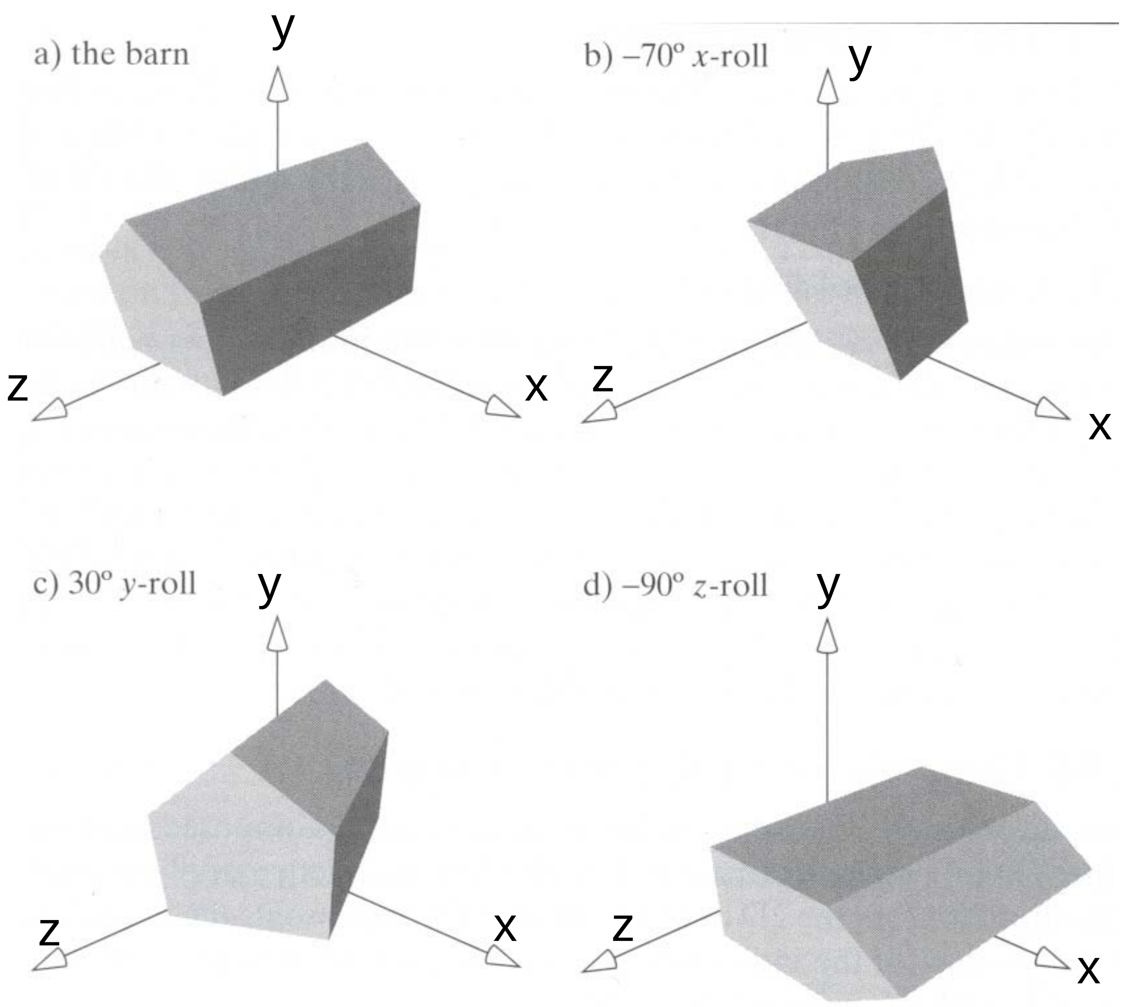


Rotating in 3D

- New terminology
  - **X-roll:** rotation about x-axis
  - **Y-roll:** rotation about y-axis
  - **Z-roll:** rotation about z-axis
- Which way is +ve rotation
  - Look in –ve direction (into +ve arrow)
  - CCW is +ve rotation



Rotating in 3D



Rotating in 3D

- For a rotation angle, β about an axis
- Define:

$$c = \cos(\beta) \qquad s = \sin(\beta)$$

x-roll or (RotateX)

$$R_x(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotating in 3D

y-roll (or RotateY)

$$R_y(\beta) = \begin{pmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rules:

- Write 1 in rotation row, column
- Write 0 in the other rows/columns
- Write c,s in rect pattern

z-roll (or RotateZ)

$$R_z(\beta) = \begin{pmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: Rotating in 3D

Question: Using **y-roll** equation, rotate *P* = (3,1,4) by 30 degrees:

Answer: c = cos(30) = 0.866, s = sin(30) = 0.5, and

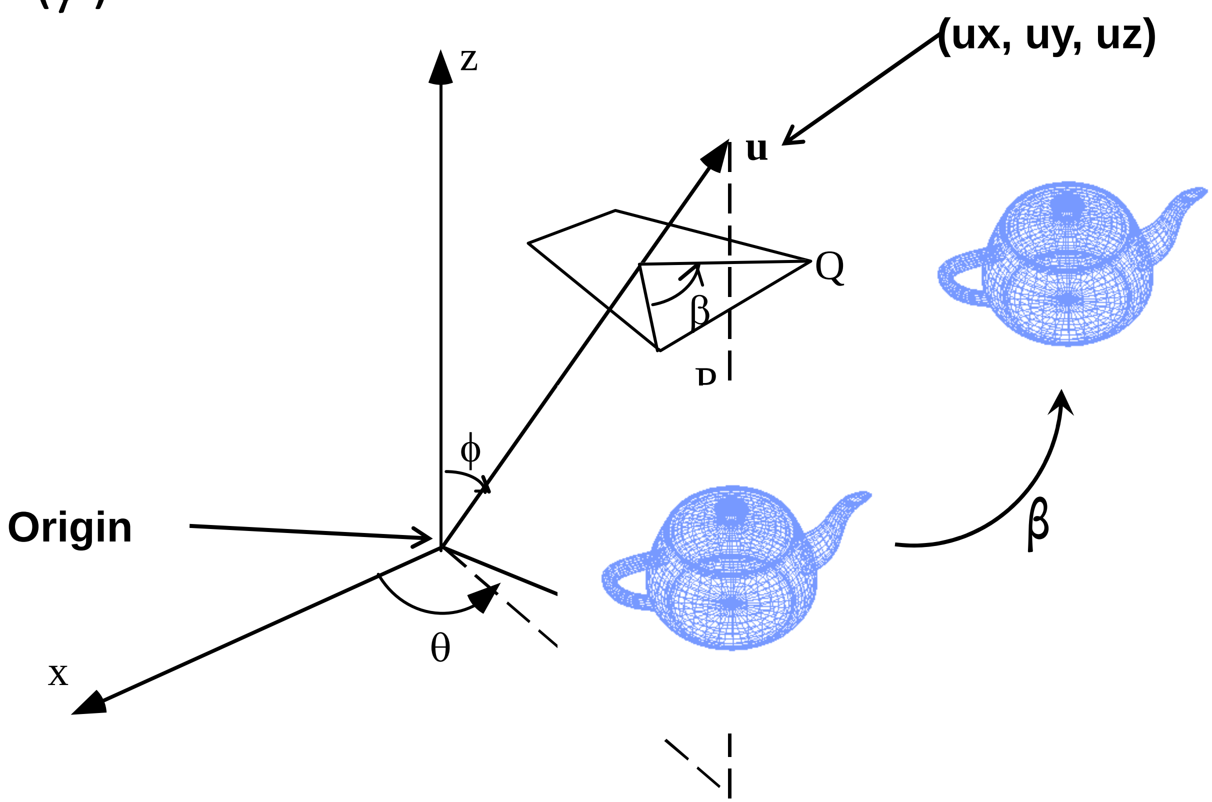
$$Q = \begin{pmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.6 \\ 1 \\ 1.964 \\ 1 \end{pmatrix}$$

Line 1: 3.c + 1.0 + 4.s + 1.0

= 3 x 0.866 + 4 x 0.5 = 4.6

3D Rotation

- **Rotate(angle, ux, uy, uz):** rotate by angle β about an **arbitrary** axis (a vector) passing through **origin** and **(ux, uy, uz)**
- **Note:** Angular position of **u** specified as azimuth (θ) and latitude (ϕ)



Approach 1: 3D Rotation About Arbitrary Axis

- Can compose arbitrary rotation as combination of:
  - X-roll (by an angle β_1)
  - Y-roll (by an angle β_2)
  - Z-roll (by an angle β_3)

$$M = R_z(\beta_3)R_y(\beta_2)R_x(\beta_1)$$

←

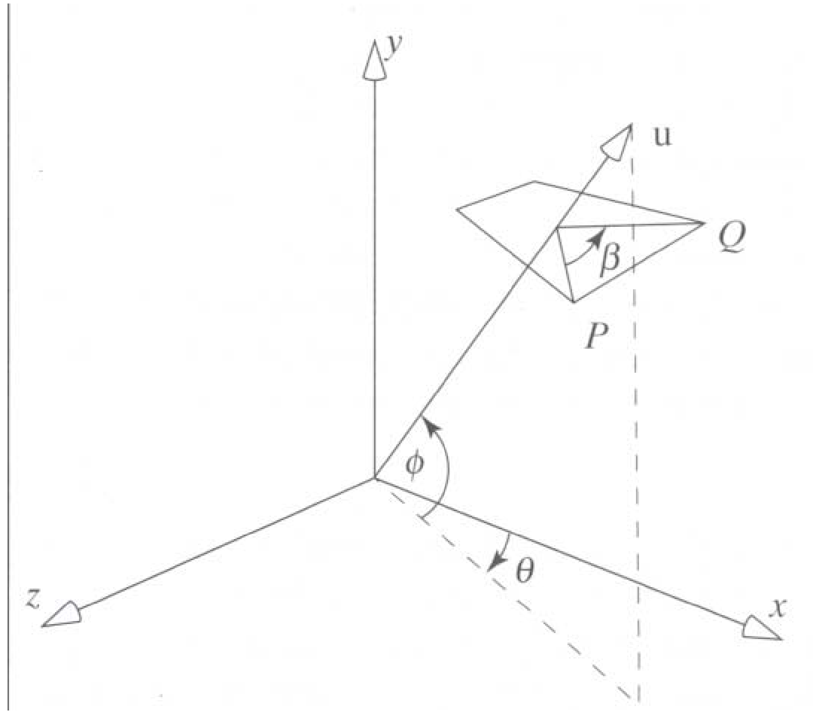
Read in reverse order

Approach 1: 3D Rotation using Euler Theorem

- **Classic:** use Euler's theorem
- **Euler's theorem:** any sequence of rotations = one rotation about some axis
- Want to rotate β about arbitrary axis **u** through origin
- Our approach:
  1. Use two rotations to align **u** and **x-axis**
  2. Do **x-roll** through angle β
  3. Negate two previous rotations to de-align **u** and **x-axis**

Approach 1: 3D Rotation using Euler Theorem

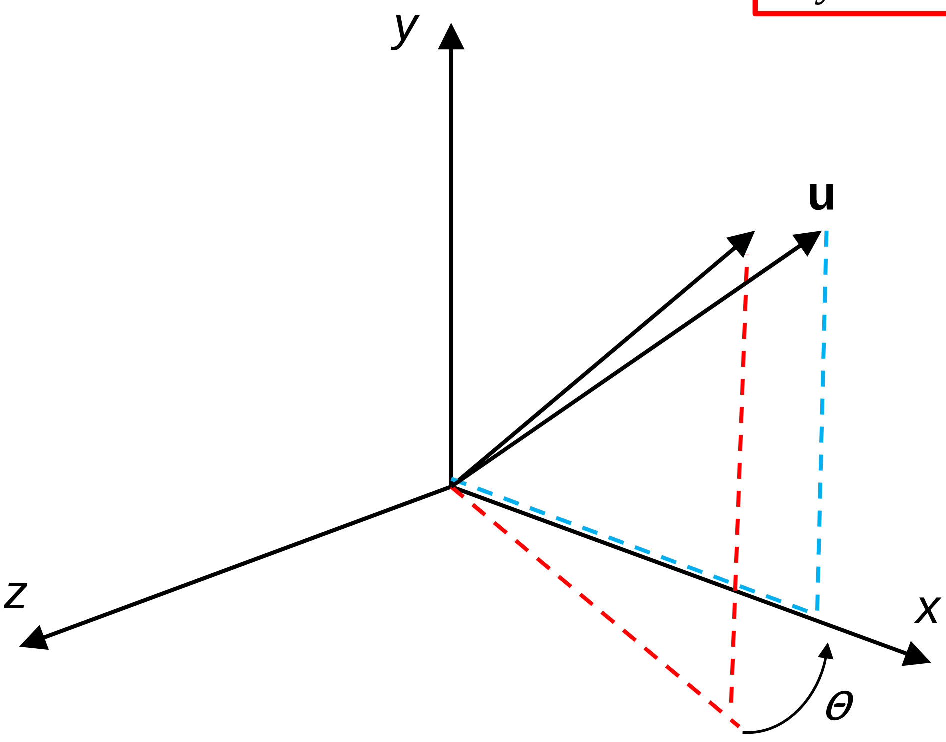
- **Note:** Angular position of **u** specified as azimuth (Θ) and latitude (ϕ)
- First try to align **u** with x axis



Approach 1: 3D Rotation using Euler Theorem

- **Step 1:** Do y-roll to line up rotation axis with x-y plane

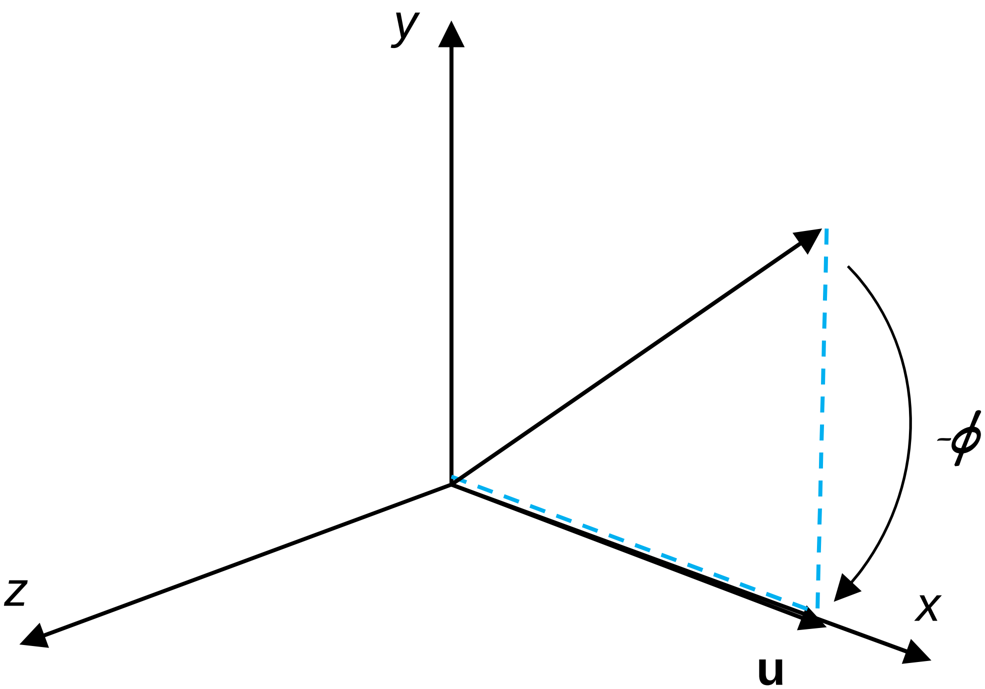
$R_y(\theta)$



Approach 1: 3D Rotation using Euler Theorem

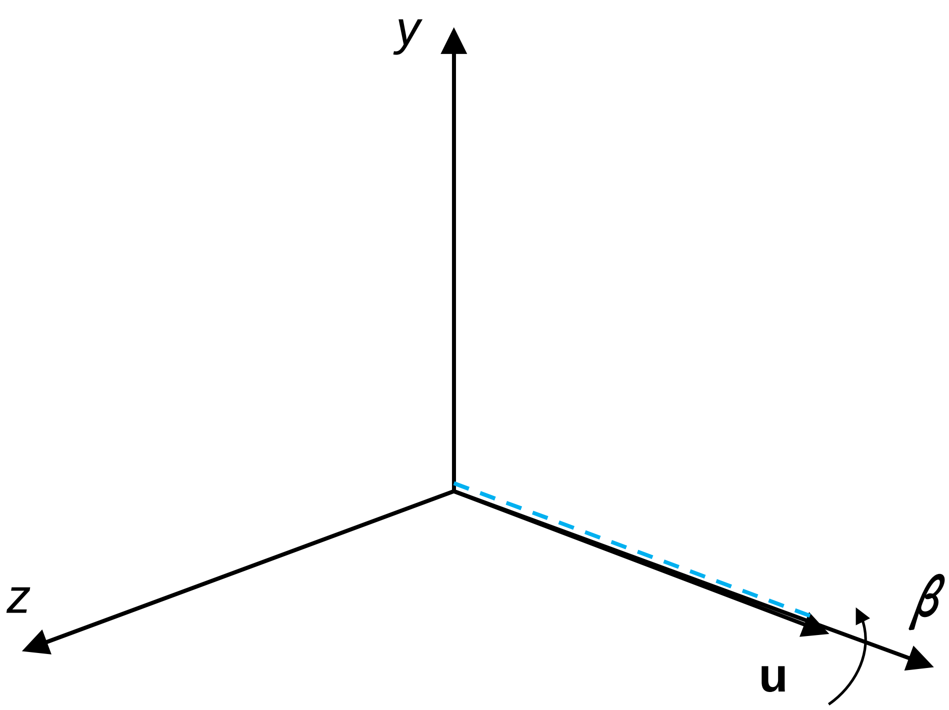
- **Step 2:** Do z-roll to line up rotation axis with x axis

$$R_z(-\phi)R_y(\theta)$$

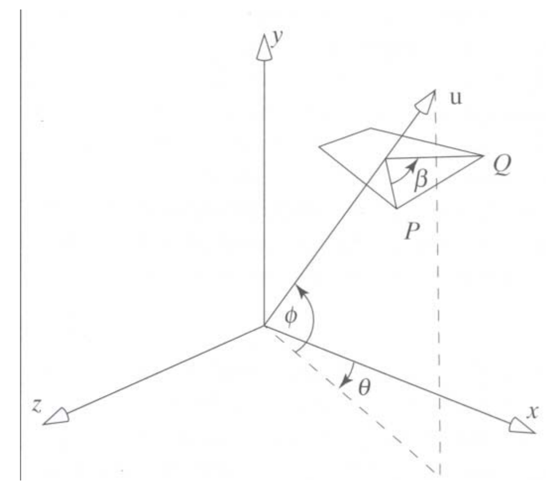


Approach 1: 3D Rotation using Euler Theorem

- **Remember:** Our goal is to do rotation by β around **u**
- But axis **u** is now lined up with x axis. So,
- **Step 3:** Do x-roll by β around axis **u**



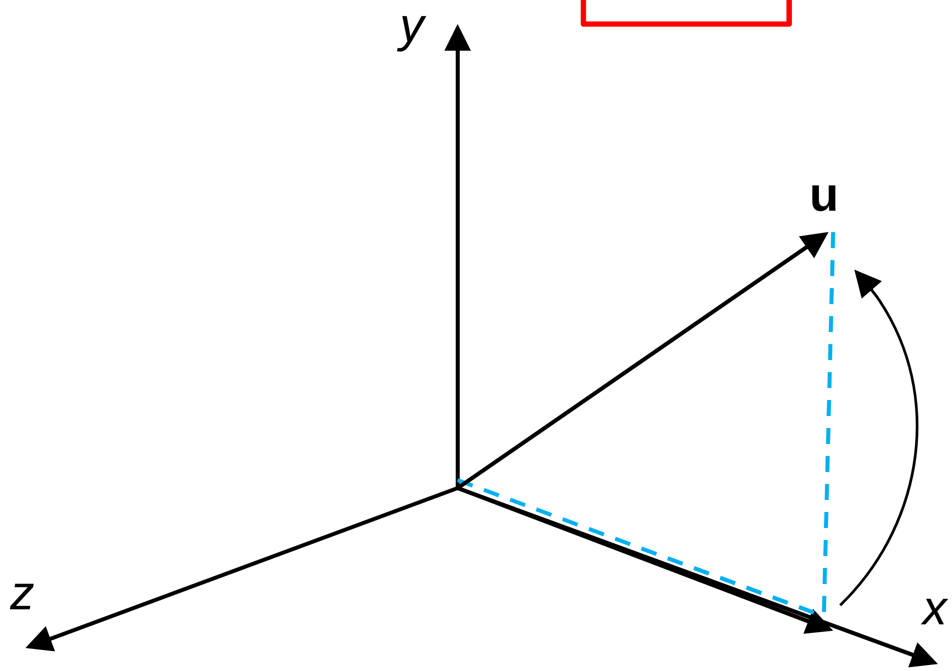
$$\boxed{R_x(\beta)}R_z(-\phi)R_y(\theta)$$



Approach 1: 3D Rotation using Euler Theorem

- Next 2 steps are to return vector **u** to original position
- **Step 4:** Do z-roll in x-y plane

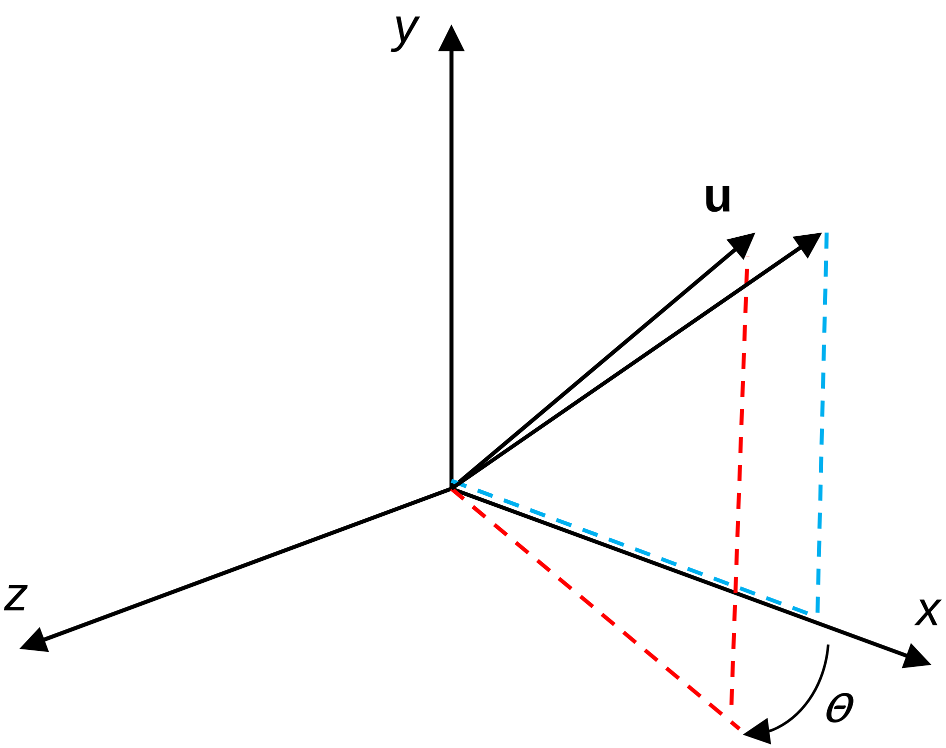
$$R_z(\phi)R_x(\beta)R_z(-\phi)R_y(\theta)$$

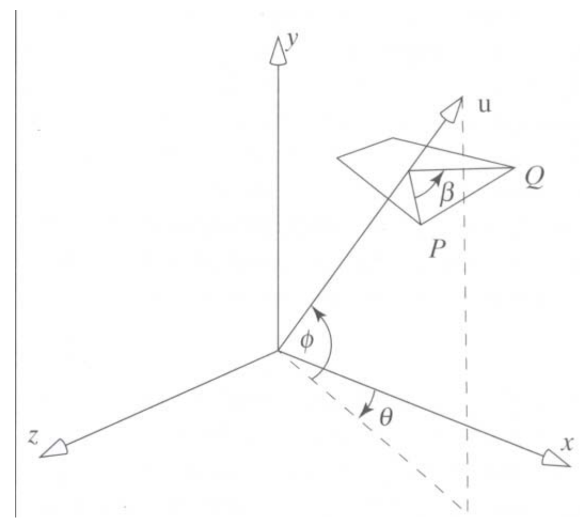


Approach 1: 3D Rotation using Euler Theorem

- **Step 5:** Do y-roll to return **u** to original position

$$R_u(\beta) = R_y(-\theta) R_z(\phi) R_x(\beta) R_z(-\phi) R_y(\theta)$$





Approach 2: Rotation using Quartenions

- Extension of imaginary numbers from 2 to 3 dimensions
- Requires 1 real and 3 imaginary components $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

- Quaternions can express rotations on sphere smoothly and efficiently

Approach 2: Rotation using Quartenions

- Derivation skipped! Check answer
- Solution has lots of symmetry

$$R(\beta)=\begin{pmatrix} c+(1-c)\mathbf{u}_x^{\ 2} & (1-c)\mathbf{u}_y\mathbf{u}_x+s\mathbf{u}_z & (1-c)\mathbf{u}_z\mathbf{u}_x+s\mathbf{u}_y & 0 \\ (1-c)\mathbf{u}_x\mathbf{u}_y+s\mathbf{u}_z & c+(1-c)\mathbf{u}_y^{\ 2} & (1-c)\mathbf{u}_z\mathbf{u}_y-s\mathbf{u}_x & 0 \\ (1-c)\mathbf{u}_x\mathbf{u}_z-s\mathbf{u}_y & (1-c)\mathbf{u}_y\mathbf{u}_z-s\mathbf{u}_x & c+(1-c)\mathbf{u}_z^{\ 2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$c=\cos(\beta) \qquad s=\sin(\beta)$$

Inverse Matrices

- Can compute inverse matrices by general formulas
- But easier to use simple geometric observations
  - Translation: $\mathbf{T}^{-1}(d_x, d_y, d_z) = \mathbf{T}(-d_x, -d_y, -d_z)$
  - Scaling: $\mathbf{S}^{-1}(s_x, s_y, s_z) = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$
  - Rotation: $\mathbf{R}^{-1}(q) = \mathbf{R}(-q)$
    - Holds for any rotation matrix

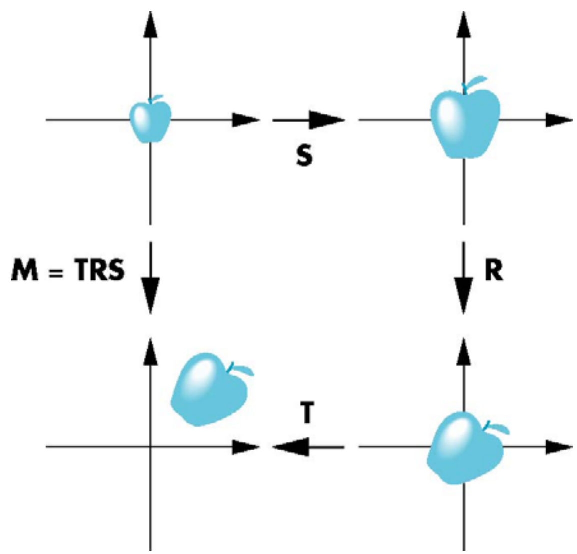
Instancing

- During modeling, often start with simple object centered at origin, aligned with axis, and unit size
- Can declare one copy of each shape in scene
- E.g. declare 1 mesh for soldier, 500 instances to create army
- Then apply *instance transformation* to its vertices to

    Scale

    Orient

    Locate



Concatenating Transformations

- Can form arbitrary affine transformation matrices by multiplying rotation, translation, and scaling matrices
- General form:

M1 X M2 X M3 X P

where M1, M2, M3 are transform matrices applied to P

- Be careful with the order!!
- For example:
  - Translate by (5,0) then rotate 60 degrees NOT same as
  - Rotate by 60 degrees then translate by (5,0)

Concatenation Order

- Note that matrix on right is first applied
- Mathematically, the following are equivalent

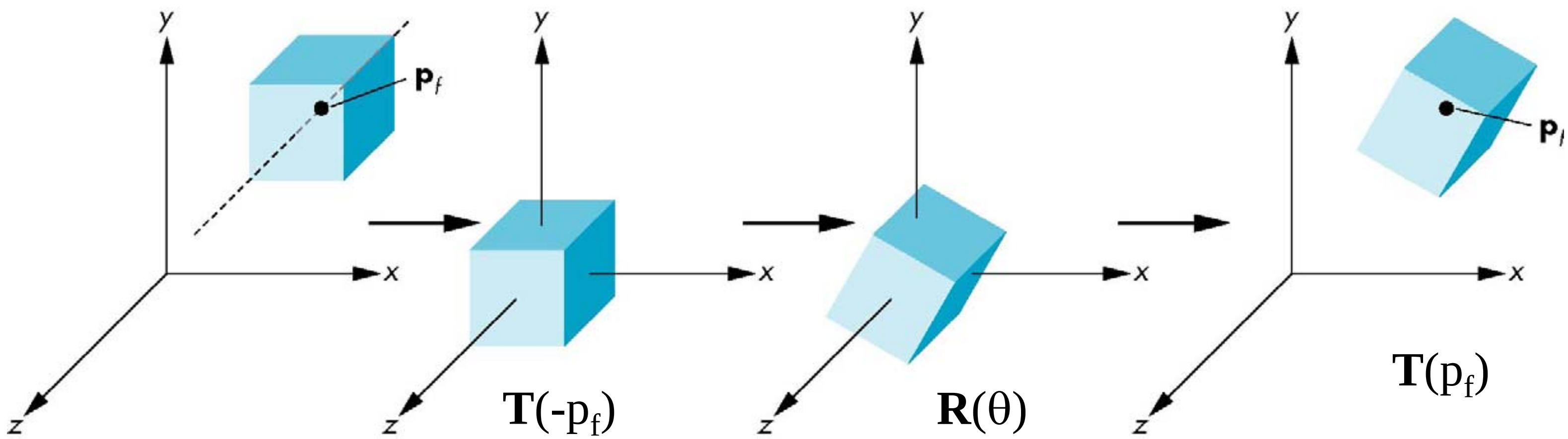
$$\mathbf{p}' = \mathbf{ABCp} = \mathbf{A}(\mathbf{B}(\mathbf{Cp}))$$

- **Efficient!!**
  - Matrix $\mathbf{M}=\mathbf{ABC}$ is composed, then multiplied by many vertices
  - Cost of forming matrix $\mathbf{M}=\mathbf{ABC}$ not significant compared to cost of multiplying $(\mathbf{ABC})\mathbf{p}$ for many vertices $\mathbf{p}$ one by one

Rotation About Arbitrary Point other than the Origin

- Default rotation matrix is about origin
- How to rotate about any arbitrary point (Not origin)?
  - Move fixed point to origin $\mathbf{T}(-p_f)$
  - Rotate $\mathbf{R}(\theta)$
  - Move fixed point back $\mathbf{T}(p_f)$

So, $\mathbf{M} = \mathbf{T}(p_f)\,\mathbf{R}(\theta)\,\mathbf{T}(-p_f)$



Scale about Arbitrary Center

- Similary, default scaling is about origin
- To scale about arbitrary point P = (Px, Py, Pz) by (Sx, Sy, Sz)
  1. Translate object by T(-Px, -Py, -Pz) so P coincides with origin
  2. Scale the object by (Sx, Sy, Sz)
  3. Translate object back: T(Px, Py, Py)
- In matrix form: T(Px,Py,Pz) (Sx, Sy, Sz) T(-Px,-Py,-Pz) * P

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & Px \\ 0 & 1 & 0 & Py \\ 0 & 0 & 1 & Pz \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -Px \\ 0 & 1 & 0 & -Py \\ 0 & 0 & 1 & -Pz \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

References

- Angel and Shreiner, Chapter 3
- Hill and Kelley, Computer Graphics Using OpenGL, 3^{rd} edition