CS 563 Advanced Topics in Computer Graphics
Monte Carlo Integration: Basic Concepts

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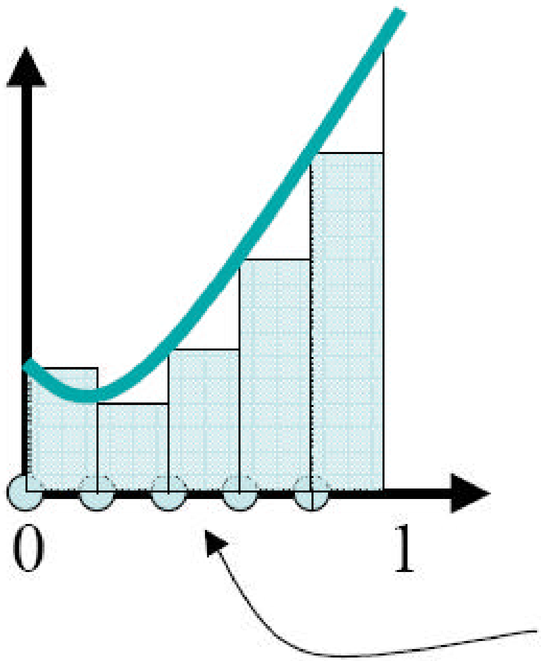
Introduction

- The integral equations generally don't have analytic solutions, so we must turn to numerical methods.

$$L_o(\mathrm{p}, ?_{\mathrm{o}}) = L_e(\mathrm{p}, ?_{\mathrm{o}}) + \int_{S^2} f(\mathrm{p}, ?_{\mathrm{o}}, ?_{\mathrm{i}}) L_i(\mathrm{p}, ?_{\mathrm{i}}) \left|\cos ?_{\mathrm{i}}\right| d?_{\mathrm{i}}$$

- Standard methods like Trapezoidal integration or Gaussian quadrature are not effective for high-dimensional and discontinuous integrals.

Simple integration

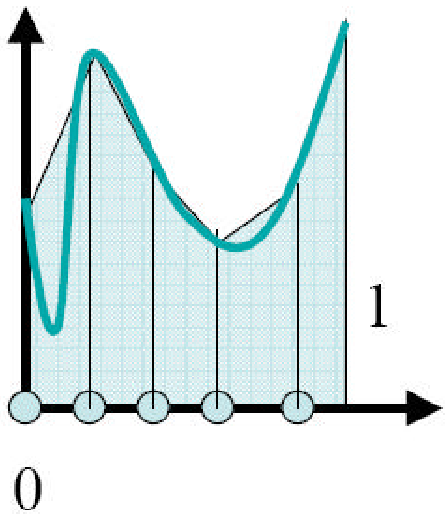


$$\int_0^1 f(x)dx \approx \sum_{i=1}^{N} f(x_i)\Delta x$$

$$= \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

$$\text{Error} = O\left(\frac{1}{N}\right)$$

Trapezoidal rule



$$\int_0^1 f(x)dx \approx \sum_{i=0}^{N-1}\left(f(x_i) + f(x_{i+1})\right)\frac{\Delta x}{2}$$

$$= \frac{1}{N}\sum_{i=1}^{N} w_i f(x_i)$$

$$w_i = \begin{cases} 0.5 & i = 0, N \\ 1 & 0 < i < N \end{cases}$$

$$\text{Error} = O\left(\frac{1}{N}\right)$$

Randomized algorithms

- *Las Vegas* v.s. *Monte Carlo*
- *Las Vegas:* gives the right answer by using randomness.
- *Monte Carlo*: gives the right answer *on the average*.
  - Results depend on random numbers used
  - Statistically likely to be close to right answer

Monte Carlo integration

- Monte Carlo integration:
  - uses sampling to estimate the values of integrals.
  - Evaluate integrand at arbitrary points,
  - *Easy to implement* and *applicable to many problems*.
- If n samples used, converges at rate $O(n^{-1/2})$.
  - To cut error by 2x, sample 4x.
- Monte Carlo images are often noisy.

Monte Carlo methods

Advantages

- Easy to implement
- Easy to think about (but be careful of statistical bias)
- Robust when used with complex integrands and domains (shapes, lights, ...)
- Efficient for high dimensional integrals
- Efficient solution method for a few selected points

Disadvantages

- Noisy
- Slow (many samples needed for convergence)

Basic concepts

- X is random variable
- Applying a function to a random variable gives another random variable, $Y=f(X)$.
- CDF (cumulative distribution function)

$$P(x) \equiv \Pr\{X \leq x\}$$

- PDF (probability density function): nonnegative, sum to 1

$$p(x) \equiv \frac{dP(x)}{dx}$$

- canonical uniform random variable ? (provided by standard library and easy to transform to other distributions)

Discrete probability distributions

Discrete events X_i with probability p_i

$$p_i \geq 0 \qquad \sum_{i=1}^{n} p_i = 1$$

Cumulative PDF

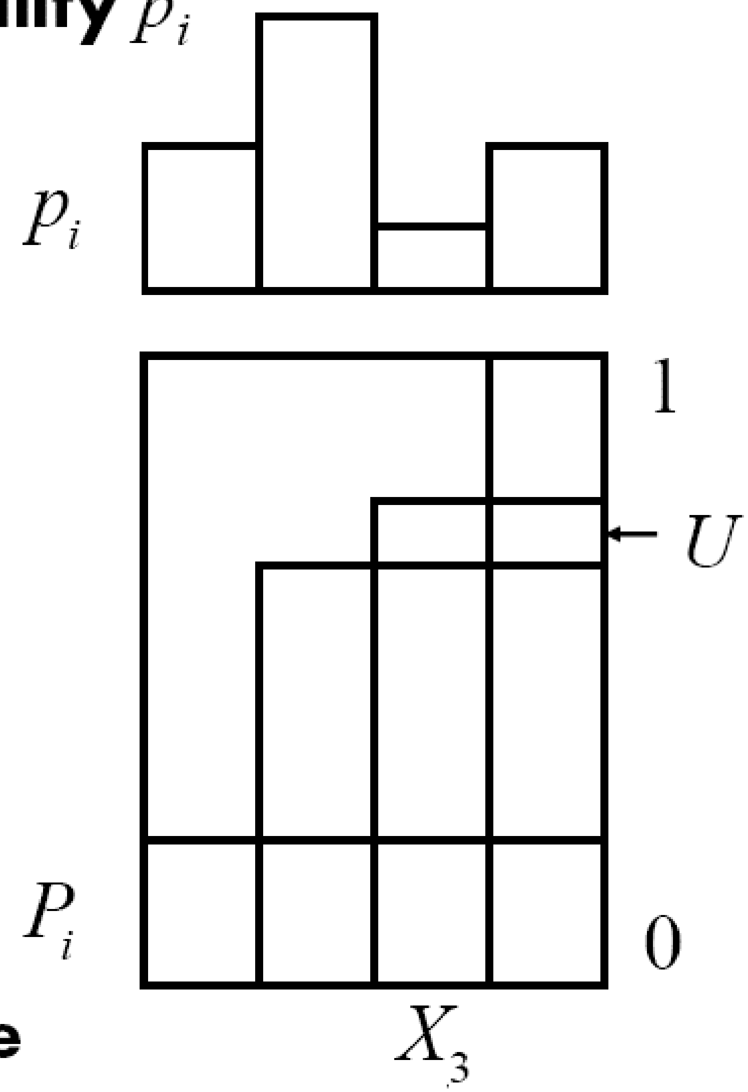
$$P_j = \sum_{i=1}^{j} p_i$$

Construction of samples

To randomly select an event,

Select X_i if $P_{i-1} < U \leq P_i$

Uniform random variable



Continuous probability distributions

PDF $p(x)$

$$p(x) \geq 0$$

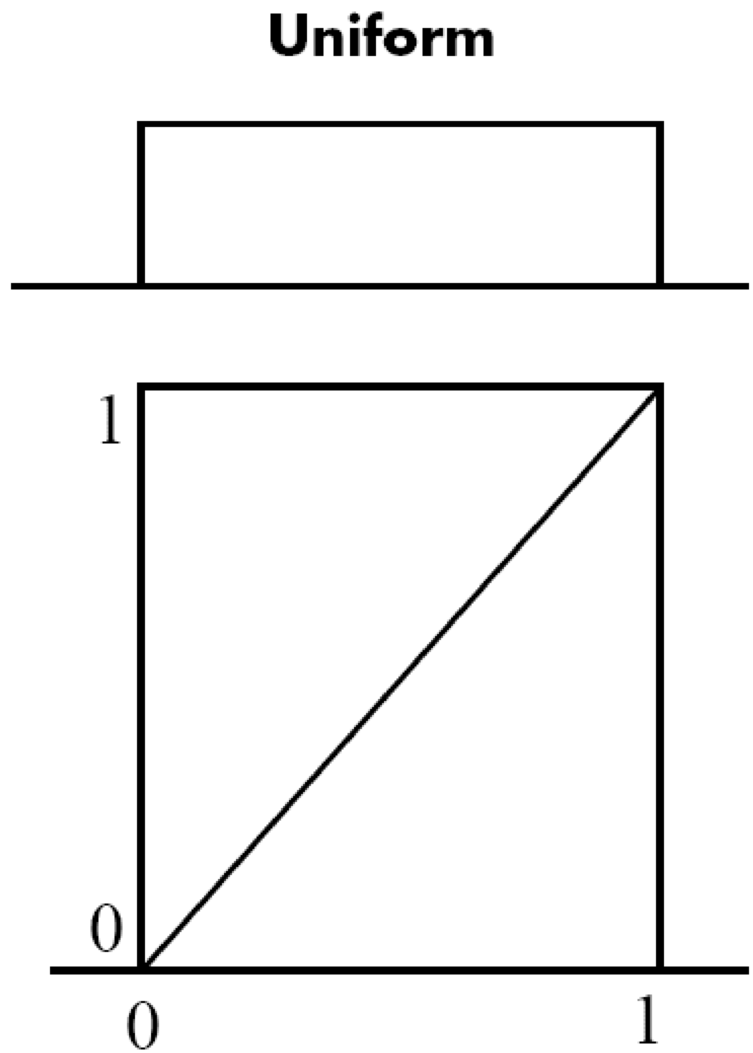
CDF $P(x)$

$$P(x) = \int_0^x p(x)dx$$

$$P(x) = \Pr(X < x) \qquad P(1) = 1$$

$$\Pr(\alpha \leq X \leq \beta) = \int_\alpha^\beta p(x)\,dx$$

$$= P(\beta) - P(\alpha)$$



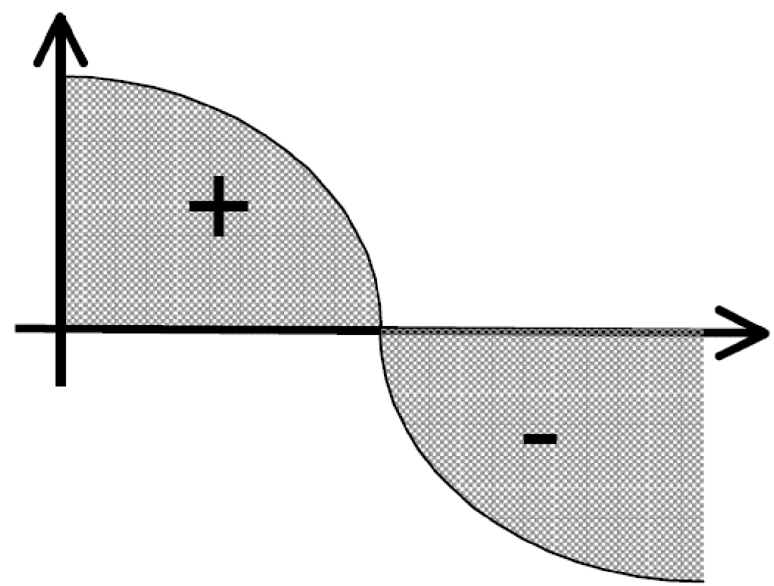
Expected values

- Average value of a function *f(x)* over some distribution of values *p(x)* over its domain *D*

$$E_p[f(x)] = \int_D f(x)p(x)dx$$

- Example: cos function over [0, p], p is uniform

$$E_p[\cos(x)] = \int_0^p \cos x \frac{1}{p} dx = 0$$



Variance

- Expected deviation from the expected value
- Fundamental concept of quantifying the error in Monte Carlo methods

$$V[f(x)] = E\left[\left(f(x) - E[f(x)]\right)^2\right]$$

Properties

$$E\left[af(x)\right] = aE\left[f(x)\right]$$

$$E\left[\sum_i f(X_i)\right] = \sum_i E\left[f(X_i)\right]$$

$$V\left[af(x)\right] = a^2 V\left[f(x)\right]$$

$$\rightarrow V\left[f(x)\right] = E\left[(f(x))^2\right] - E\left[f(x)\right]^2$$

Monte Carlo estimator

- Assume that we want to evaluate the integral of $f(x)$ over $[a,b]$
- Given a uniform random variable X_i over $[a,b]$, Monte Carlo estimator says that the expected value $E[F_N]$ of the estimator F_N equals the integral

$$F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i)$$

$$\begin{aligned} E[F_N] &= E\left[\frac{b-a}{N}\sum_{i=1}^{N} f(X_i)\right] \\ &= \frac{b-a}{N}\sum_{i=1}^{N} E[f(X_i)] \\ &= \frac{b-a}{N}\sum_{i=1}^{N}\int_a^b f(x)p(x)dx \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_a^b f(x)dx \\ &= \int_a^b f(x)dx \end{aligned}$$

General Monte Carlo estimator

- Given a random variable X drawn from an arbitrary PDF p(x), then the estimator is

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$

$$E[F_N] = E\left[\frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}\right] = \frac{1}{N}\sum_{i=1}^{N}\int_a^b \frac{f(x)}{p(x)}p(x)dx = \int_a^b f(x)dx$$

- Although the converge rate of MC estimator is $O(N^{1/2})$, slower than other integral methods, its converge rate is independent of the dimension, making it the only practical method for high dimensional integral

Choosing samples

- How to sample an arbitrary distribution from a variable of uniform distribution?
  - Inversion
  - Rejection
  - Transform

Inversion method

Cumulative probability distribution function

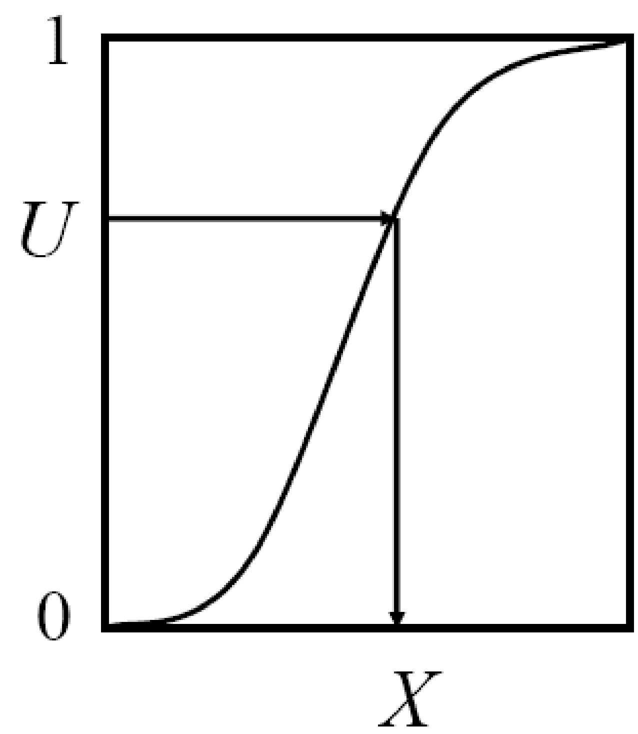
$$P(x) = \Pr(X < x)$$

Construction of samples

Solve for $X = P^{-1}(U)$

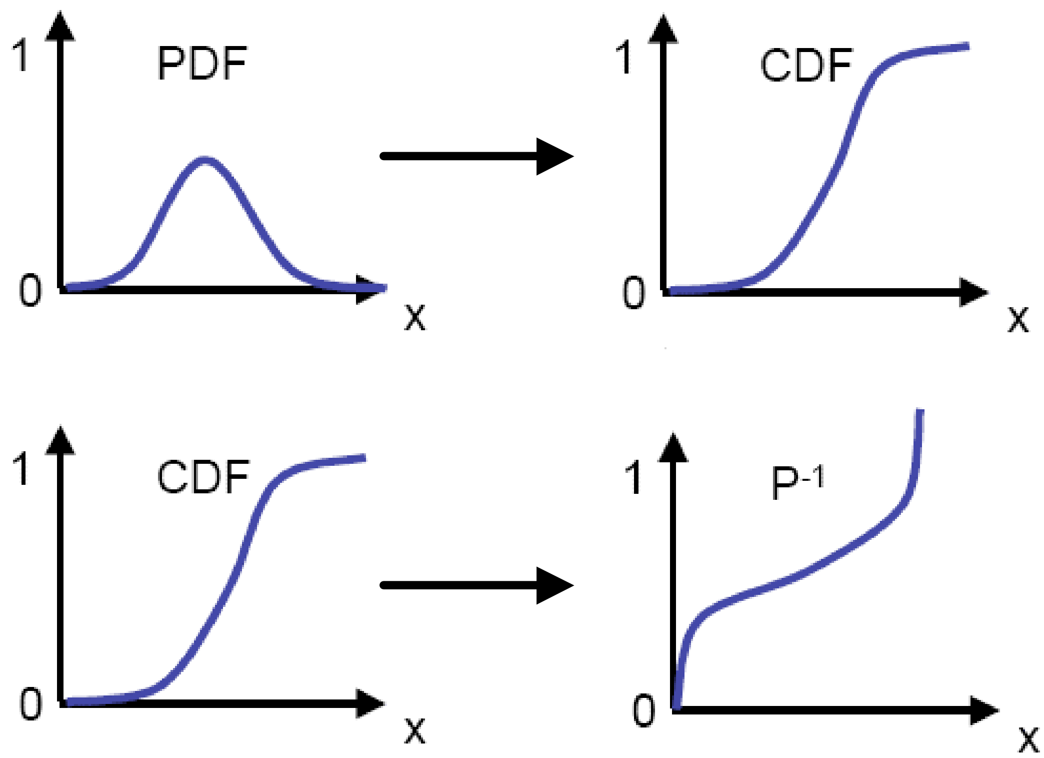
Must know:

1. **The integral of** $p(x)$
2. **The inverse function** $P^{-1}(x)$



Inversion method

- Compute CDF $P(x)$
- Compute $P^{-1}(x)$
- Obtain ?
- Compute $X_i = P^{-1}(?\ )$



Example: exponential distribution

$p(x) = ce^{-ax}$, for example, Blinn's Fresnel term

$$\int_0^\infty ce^{-ax}\,dx = 1 \longrightarrow c = a$$

- Compute CDF P(x)

$$P(x) = \int_0^x ae^{-as}\,ds = 1 - e^{-ax}$$

- Compute $P^{-1}(x)$

$$P^{-1}(x) = -\frac{1}{a}\ln(1-x)$$

- Obtain ?
- Compute $X_i = P^{-1}(?\ )$

$$X = -\frac{1}{a}\ln(1-\xi) = -\frac{1}{a}\ln\xi$$

Example: power function

Assume

$$\int_0^1 x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$$

$$p(x) = (n+1)x^n$$

$$P(x) = x^{n+1}$$

$$X \sim p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}$$

Trick

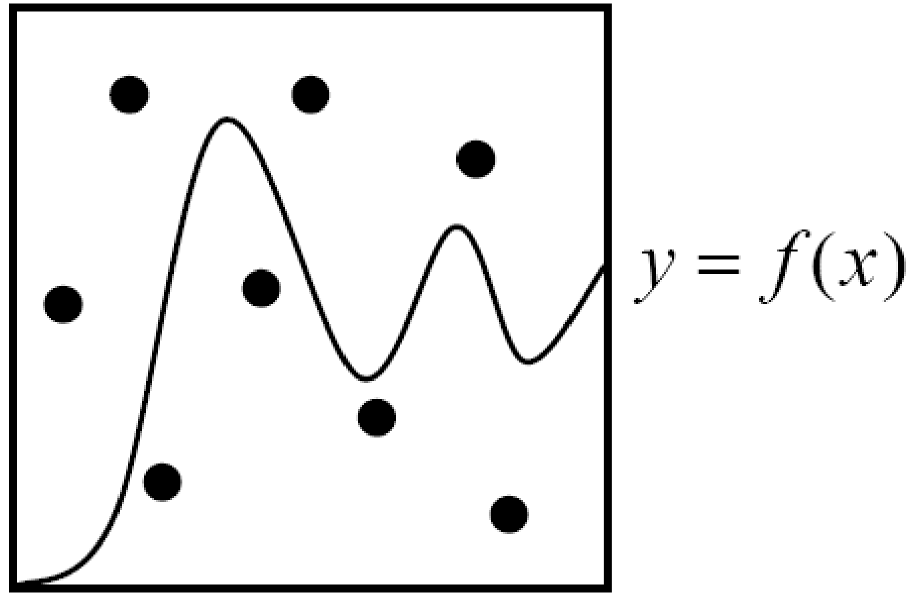
$$Y = \max(U_1, U_2, \cdots, U_n, U_{n+1})$$

$$\Pr(Y < x) = \prod_{i=1}^{n+1} \Pr(U < x) = x^{n+1}$$

Rejection method

$$I = \int_0^1 f(x)\,dx$$

$$= \iint_{y<f(x)} dx\,dy$$



Algorithm

Pick U_1 **and** U_2

Accept U_1 **if** $U_2 < f(U_1)$

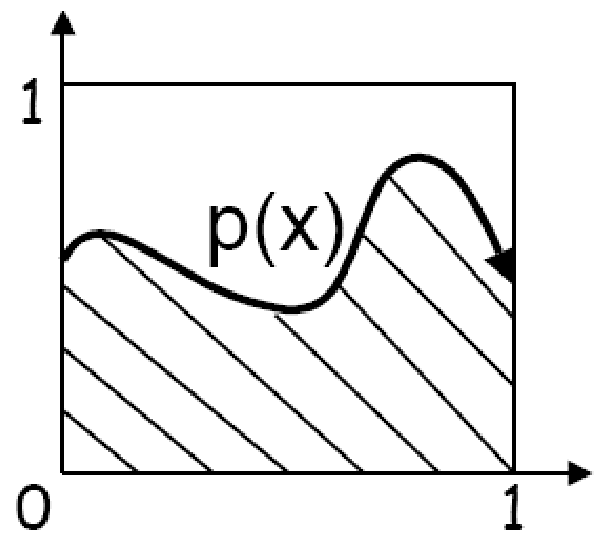
Wasteful? Efficiency = Area / Area of rectangle

Rejection method

- Sometimes, we can't integrate into CDF or invert CDF
- Rejection method is a fart-throwing method without performing the above steps

1. Find q(x) so that p(x)<cq(x)
2. Dart throwing
   a. Choose a pair (X, ?), where X is sampled from q(x)
   b. If (? <p(X)/cq(X)) return X

- Essentially, we pick a point (X, ? cq(X)). If it lies beneath p(X) then we are fine.



Example: sampling a unit sphere

```
void RejectionSampleDisk(float *x, float *y) {
  float sx, sy;
  do {
    sx = 1.f -2.f * RandomFloat();
    sy = 1.f -2.f * RandomFloat();
  } while (sx*sx + sy*sy > 1.f)
  *x = sx;  *y = sy;
}
```

p /4~ 78.5% good samples, gets worse in higher dimensions, for example, for sphere, p /6~ 52.3%

Transforming between distributions

- Transform a random variable X from distribution $p_x(x)$ to a random variable Y with distribution $p_y(x)$
- Y=y(X), y is one-to-one, i.e. monotonic
- Hence,
- PDF:

$$P_y(y) = \Pr\{Y \le y(x)\} = \Pr\{X \le x\} = P_x(x)$$

$$\frac{dP_y(y)}{dx} = \frac{dP_x(x)}{dx}$$

$$p_y(y)\frac{dy}{dx} = \frac{dP_y(y)}{dy}\frac{dy}{dx} \qquad p_x(x)$$

$$p_y(y) = \left(\frac{dy}{dx}\right)^{-1} p_x(x)$$

Example

$$p_x(x) = 2x$$

$$Y = \sin X$$

$$p_y(y) = (\cos x)^{-1} p_x(x) = \frac{2x}{\cos x} = \frac{2\sin^{-1} y}{\sqrt{1-y^2}}$$

- **Transform:** Given X with $p_x(x)$ and $p_y(y)$, try to use X to generate Y.

Multiple dimensions

- Easily generalized - using the Jacobian of Y=T(X)

$$p_y(T(x)) = \left|J_T(x)\right|^{-1} p_x(x)$$

- Example - polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$J_T(x) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$p(r,\theta) = \left|J_T\right|^{-1} p(x,y) = r p(x,y)$$

Multiple dimensions

- Spherical coordinates:

$$p(r,\theta,\phi) = r^2 \sin\theta p(x,y,z)$$

- Now looking at spherical directions:
- We want to solid angle to be uniformly distributed $d\omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$
- Hence the density in terms of ϕ and θ:

$$p(\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi = p(\omega)\mathrm{d}\omega$$

$$p(\theta,\phi) = \sin\theta p(\omega)$$

Multidimensional sampling

- Separable case - independently sample X from p_x and Y from p_y: $p(x,y) = p_x(x)p_y(y)$
- Often times this is not possible - compute the marginal density function p(x) first:

$$p(x) = \int p(x,y)dy$$

- Then compute conditional density function (p of y given x) $p(y|x) = \frac{p(x,y)}{p(x)}$
- Use 1D sampling with p(x) and p(y|x)

Sampling a hemisphere

- Uniformly, I.e. p(ω) = c

$$1 = \int_{H^2} p(\omega) \qquad c = \frac{1}{2\pi}$$

- Sampling θ first:

$$p(\theta) = \int_0^{2\pi} p(\theta,\phi)\,d\phi = \int_0^{2\pi} \frac{\sin\theta}{2\pi}\,d\phi = \sin\theta$$

- Now sampling in ϕ:

$$p(\phi \mid \theta) = \frac{p(\theta,\phi)}{p(\theta)} = \frac{1}{2\pi}$$

- Note:

$$p(q,f) = \sin q \,/\, 2p$$

Sampling a hemisphere

- Now we use inversion technique in order to sample the PDF's:

$$P(\theta) = \int_0^\alpha \sin\alpha d\alpha = 1 - \cos\theta$$

$$P(\phi | \theta) = \int_0^\alpha \frac{1}{2\pi} d\alpha = \frac{\phi}{2\pi}$$

- Inverting these:

$$\theta = \cos^{-1}\xi_1$$

$$\phi = 2\pi\xi_2$$

Sampling a hemisphere

- Converting these to Cartesian coords:

$$\theta = \cos^{-1} \xi_1$$

$$\phi = 2\pi\xi_2$$

$$x = \sin\theta\cos\phi = \cos(2\pi\xi_2)\sqrt{1-\xi_1^2}$$

$$y = \sin\theta\sin\phi = \sin(2\pi\xi_2)\sqrt{1-\xi_1^2}$$

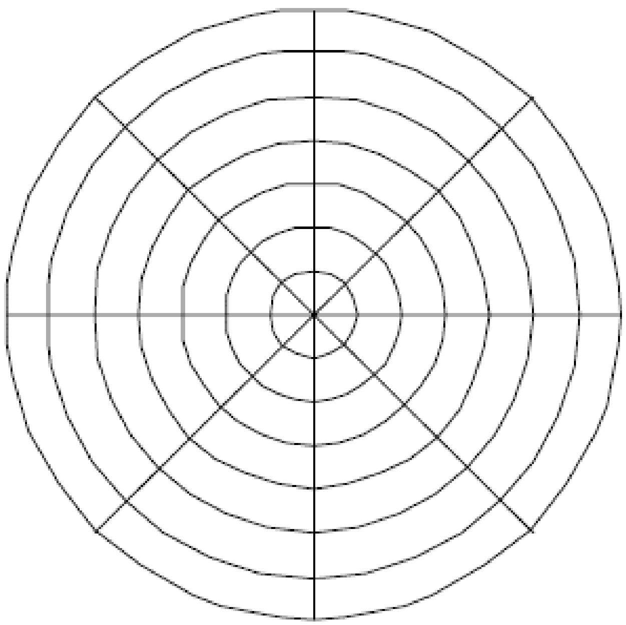
$$z = \cos\theta = \xi_1$$

- Similar derivation for a full sphere

Sampling a disk

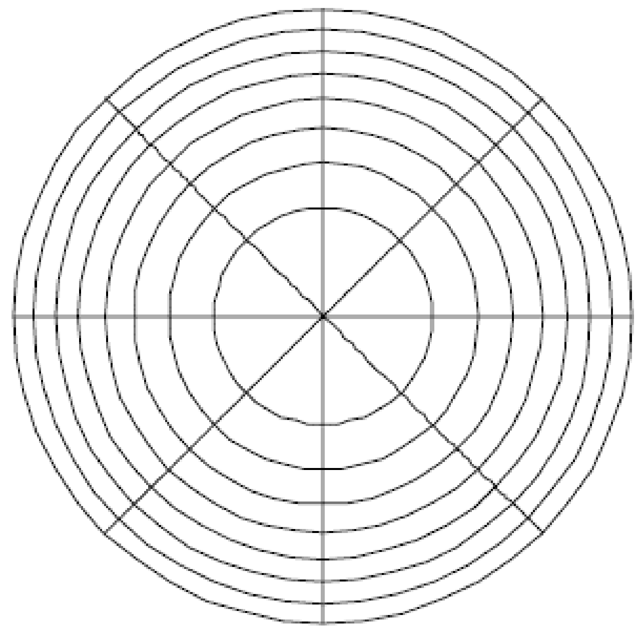
- Sampling disk is similar except **note**

WRONG ≠ Equi-Areal



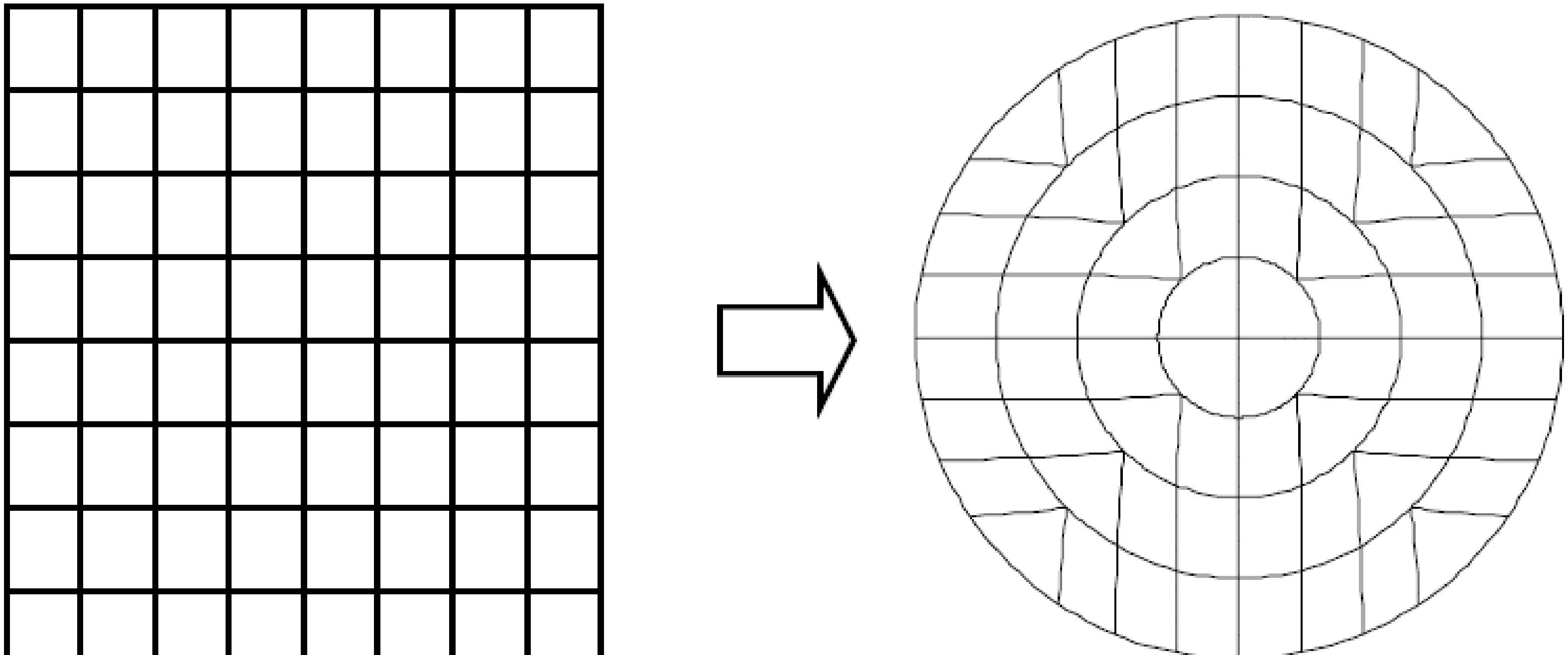
$$\theta = 2\pi U_1$$
$$r = U_2$$

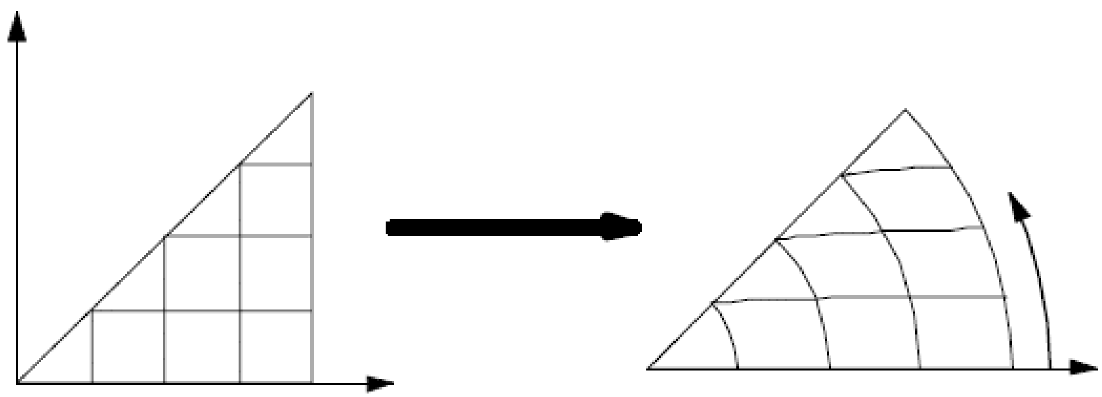
RIGHT = Equi-Areal



$$\theta = 2\pi U_1$$
$$r = \sqrt{U_2}$$

Shirley's mapping





$$r = U_1$$

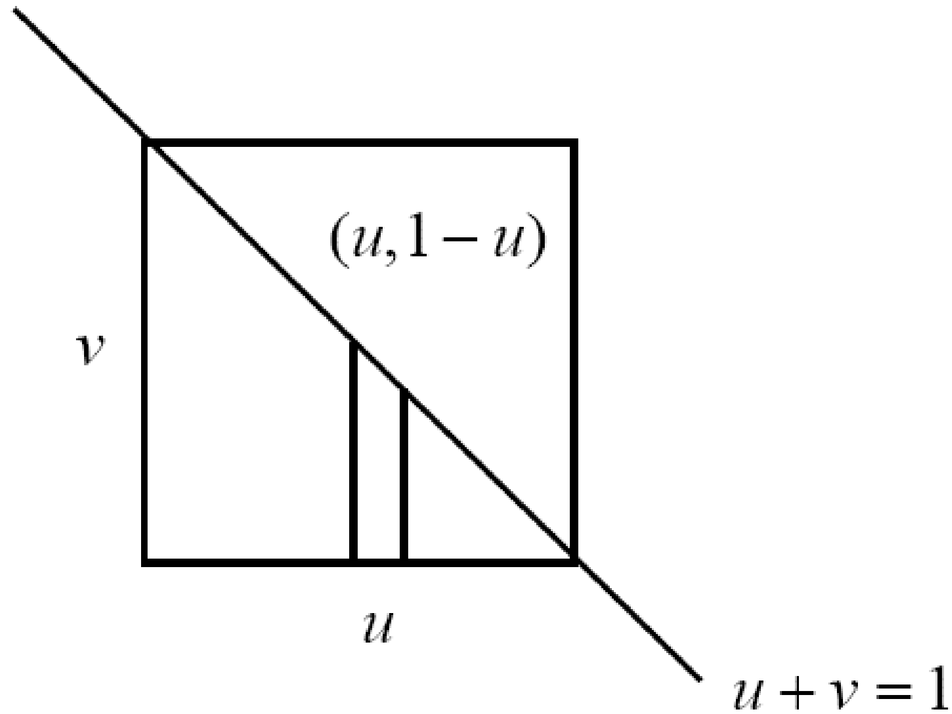
$$\theta = \frac{\pi}{4}\frac{U_2}{U_1}$$

Sampling a triangle

$$u \geq 0$$

$$v \geq 0$$

$$u + v \leq 1$$



$$A = \int_0^1 \int_0^{1-u} dv\, du = \int_0^1 (1-u)\, du = -\frac{(1-u)^2}{2}\bigg|_0^1 = \frac{1}{2}$$

$$p(u,v) = 2$$

Sampling a triangle

Here u and v are not independent! $p(u,v)=2$

Conditional probability

$$p(u) \equiv \int p(u,v)\,dv \qquad p(u\mid v) \equiv \frac{p(u,v)}{p(u)}$$

$$p(u) = 2\int_0^{1-u} dv = 2(1-u)$$

$$u_0 = 1-\sqrt{U_1}$$

$$P(u_0) = \int_0^{u_0} 2(1-u)\,du = (1-u_0)^2$$

$$p(v\mid u) = \frac{1}{(1-u)}$$

$$v_0 = \sqrt{U_1}U_2$$

$$P(v_0\mid u_0) = \int_0^{v_o} p(v\mid u_0)\,dv = \int_0^{v_o} \frac{1}{(1-u_0)}dv = \frac{v_0}{(1-u_0)}$$

Cosine weighted hemisphere

$$p(w) \propto \cos q$$

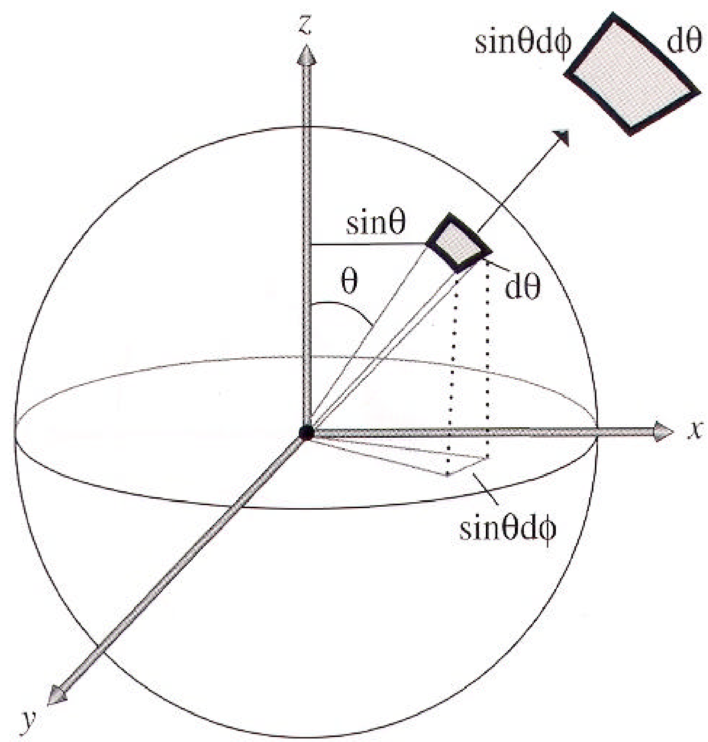
$$1 = \int_{H^2} p(w)\,dw$$

$$1 = \int_0^{2p} \int_0^{\frac{p}{2}} c \cos q \sin q \, dq \, df$$

$$1 = c2p \int_0^{\frac{p}{2}} \cos q \sin q \, dq$$

$$c = \frac{1}{p}$$

$$p(q, f) = \frac{1}{p} \cos q \sin q$$



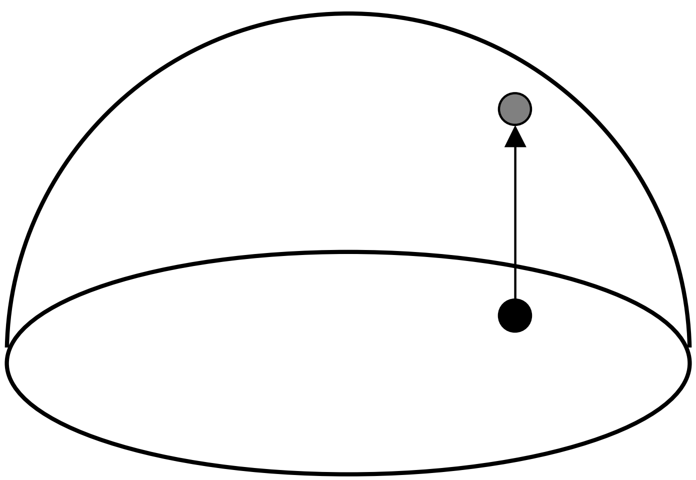
$$dw = \sin q \, dq \, df$$

- Useful for cosine weighted functions

Cosine weighted hemisphere

- Malley's method: uniformly generates points on the unit disk and then generates directions by projecting them up to the hemisphere above it.

```
Vector CosineSampleHemisphere(float u1,float u2){
  Vector ret;
  ConcentricSampleDisk(u1, u2, &ret.x, &ret.y);
  ret.z = sqrtf(max(0.f,1.f - ret.x*ret.x -
                            ret.y*ret.y));
  return ret;
}
```



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