CS2223 Algorithms. B Term 2013. **Homework 5 Solutions**

Solutions by Artem Gritsenko, Ahmedul Kabir, Yun Lu, and Prof. Ruiz

Problem I. By Artem Gritsenko, Yun Lu, and Prof. Ruiz

a. Algorithm.

KNAPSACK $(n, W, w_1, \dots, w_n, v_1, \dots, v_n)$

\# W is the knapsack capacity
\# n is the number of items that can be packed in the knapsack
\# item i has weight w_i and value v_i
\# Assume that $w_1 \leq w_2 \leq \cdots \leq w_n$ and $v_1 \geq v_2 \geq \cdots \geq v_n$
\# We'll use two variables: a set $\boldsymbol{S}$ that contains the items selected to to be pack in the
\# knapsack, and $\boldsymbol{w}$ which is the sum of the weights of the items in S.

$w = 0$

$S = \emptyset$

$i = 1$

while $(i \leq n)$ and $(w + w_i \leq W)$

$w = w + w_i$

$S = S \cup \{i\}$

$i = i + 1$

return S

b. **Complexity analysis.**

The algorithm runs in linear time in the number of items (= n). In the worst case, it will traverse the full list of n items in the while loop. Processing each individual item (loop body) takes constant time. Hence, $T(n) = O(n)$.

c. **Algorithm Correctness**.

First we show that it terminates. The while loop considers each item at most once, and hence the algorithm terminates after at most n iterations.

Now we prove that the algorithm terminates with a correct answer. That is, that the set S output by the algorithm is an optimal solution. In other words, we need to prove that:

1. S is a solution: The sum of the weights of the items in S is less than or equal to W. This is true because an item i is added to S only if the current weight of S together with w_i doesn't exceed W.

2. S is an optimal solution: For any other solution S', the sum of the values of the items in S is greater than or equal to the sum of the values of the items in S'.
   Let's prove that the set S output by the algorithm is optimal using a proof by contradiction. Assume that S is not optimal. Then there is another set O which is an optimal solution. Let's sort the items in S and the items in O in increasing order by weight (= decreasing order by value).
   $S = \{i_1, i_2, \dots, i_k\}$
   $O = \{j_1, j_2, \dots, j_m\}$
   Let $r \geq 0$ be the first index in which the two sets differ. That is,
   $S = \{i_1, i_2, \dots, i_{i-1}, i_r, i_{r+1}, \dots, i_k\}$
   $O = \{j_1, j_2, \dots, j_{i-1}, j_r, j_{r+1}, \dots, j_m\}$
   with $i_1 = j_1,\ i_2 = j_2, \dots, i_{i-1} = j_{i-1}$, and $i_r \neq j_r$.
   Since our algorithm chose i_r in such a way that i_r is the item with smaller weight and higher value among all items not in $i_1, \dots, i_{r-1}$, then:
   - the weight of i_r is less than or equal to the weight of j_r, and
   - the value of i_r is greater than or equal to the value of j_r.

   Then we use the *exchange argument:* we can replace j_r with i_r in O, without exceeding the weight limit, and keeping the value of O the same or increasing it. It the value of O increases, then this yields a contradiction as we assumed that O was optimal. If the value of O doesn't increase, then we repeat the same exchange argument with the next item, the $(r+1)^{st}$ item. We continue to do so until we transform O into a solution with higher value, which would contradict the assumption that it was optimal; or we make it equal to S without increasing its weight, which would prove that S is an optimal solution too.

Problem II. By Ahmedul Kabir and Prof. Ruiz

a. Pseudocode for Offline Caching problem

```
FurthestInFuture(<r_1, r_2, … r_n>, k):
    for i = 1 to n
        if r_i is in cache:
           then print "Cache Hit"
        else
            print "Cache Miss"
            if cache is not full
                Add r_i to cache
            else (if cache is full)
                // Find element in cache with furthest distance
                dist_furthest = 0
                for j = 1 to k
                    p = i + 1 // p is an index on the list of requests
                    Keep increasing p until r_p == cache[j]
                    if p reaches end of sequence without finding r_p
                        then
                            dist_j = infinity
                            furthest = j
                            break from the inner for loop
                    else
                        dist_j = p - i
                    if dist_j > dist_furthest
                        then
                            furthest = j
                            dist_furthest = dist_j

                Evict Cache[furthest]
                Add r_i to cache
```

Kabir's Python implementation of this pseudo-code is available at:
http://web.cs.wpi.edu/~cs2223/b13/HW/HW5/Solutions

Running time of the algorithm

The outer loop runs n times, and the inner loop k times. For each iteration of the inner loop, we may have to traverse (n – i) items at worst. So the running time is

$k\sum_{i=1}^{n}(n-i) = \frac{kn(n-1)}{2} = O(kn^2)$.

b. The off-line caching problem exhibits optimal substructure.

A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solutions to subproblems.

Let $R = \{r_1, r_2, \dots, r_n\}$ be a list of requests, and k the cache of size. Let also $E = \{e_1, e_2, \dots, e_n\}$ be a list of evictions, where each e_i is either "cache hit", if r_i is in the cache at time i, or the name of the cache element to be evicted at time i because of a cache miss occurred at that time. Let's assume that E is optimal for R and k. That is, E contains the minimum possible number of evictions (= cache misses) for the list of requests R and a cache of size k.

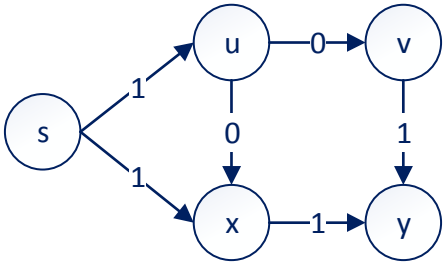
In order to show that the off-line caching problem exhibits optimal substructure , we need to show that any subsequence of $E, E_i = \{e_i, e_{i+1}, \dots, e_n\}$ is optimal for the sublist of requests $R_i = \{r_i, r_{i+1}, \dots, r_n\}$ of R starting with the cache content at time i. It suffices to show that this is true for i equal to the time when the first cache miss occurred, and then apply the same argument for other cache misses afterwards. So let $i, 1 \leq i \leq n$, be the first time when a cache miss occurred. Assume by way of contradiction that the solution $E_i = \{e_i, e_{i+1}, \dots, e_n\}$ is NOT optimal for the sublist of requests $R_i = \{r_i, r_{i+1}, \dots, r_n\}$ starting with the cache content at time i right when the cache miss was detected. Hence, there must be another solution $O = \{o_i, o_{i+1}, \dots, o_n\}$ which incurs in fewer cache misses on $R_i = \{r_i, r_{i+1}, \dots, r_n\}$ than E_i, starting with the same cache content that E_i does at time i. In that case, the list of evictions $New = \{e_1, e_2, \dots, e_{i-1}, o_i, o_{i+1}, \dots, o_n\}$ is a solution for R with fewer cache misses than E. This is a contradiction, as E is optimal for R. Hence, as claimed, E_i is optimal for R_i.

c. The furthest-in-future strategy produces the minimum possible number of cache misses. See a proof of this fact in the slides of Chapter 4 of Jon Kleinberg's and Éva Tardos' Algorithm Design textbook. Available at Prof. Kevin Wayne's Algorithms Course Webpage at Princeton University. See Greedy Algorithms Part I slides 34-43 at http://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf .

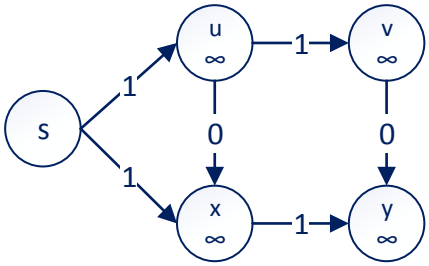
Problem III. By Yun Lu.

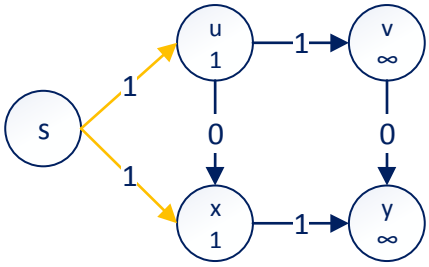
The key idea of this problem is to find a graph that has at least two optimal paths, which of course will have the same weight. Furthermore, we'll construct these two paths in such a way that Dijkstra's algorithm will relax the edges of one of the paths in the order in which they occur in the path, but will not do so for the other path.

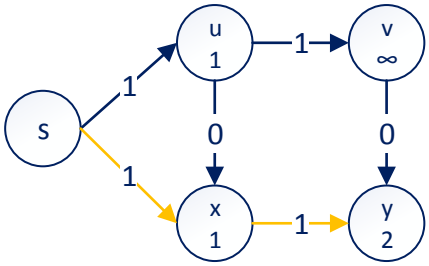
Consider the following example G= (V,E) with V={s, u, v, x, y} and E={(s,x),(s,u),(u,v),(u,x),(v,y),(x,y)}, where the weights of these edges are given below:

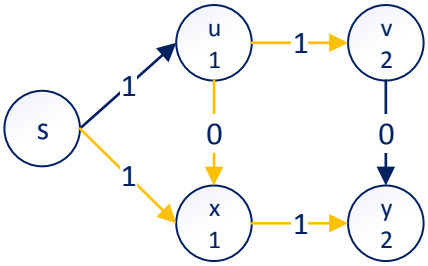


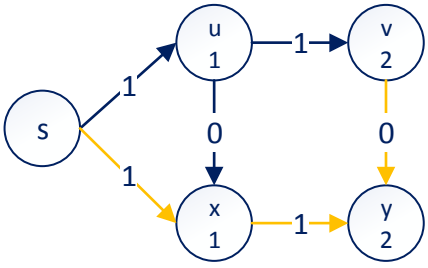
Applying Dijkstra's algorithm to the above graph:

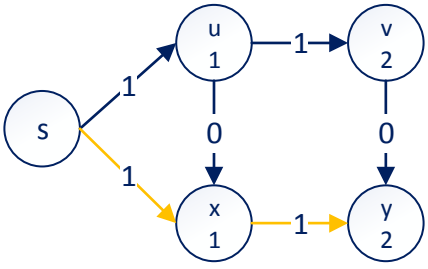












In the execution of Dijkstra's algorithm above, the edges are relaxed (in yellow) in the following order: (s,x), (s,u), (x,y), (u,x), (u,v), (v,y).

Note that there are three paths from s to y: $\begin{cases} s \to u \to v \to y \\ s \to u \to x \to y \\ s \to x \to y \end{cases}$ all of them with weight 2, hence all three paths are optimal. The edges in the path $s \to u \to x \to y$ are relaxed out of order since (x,y) is relaxed before (u,x).