Computer Graphics (CS 4731)
Lecture 12: Linear Algebra for Graphics (Points, Scalars, Vectors)

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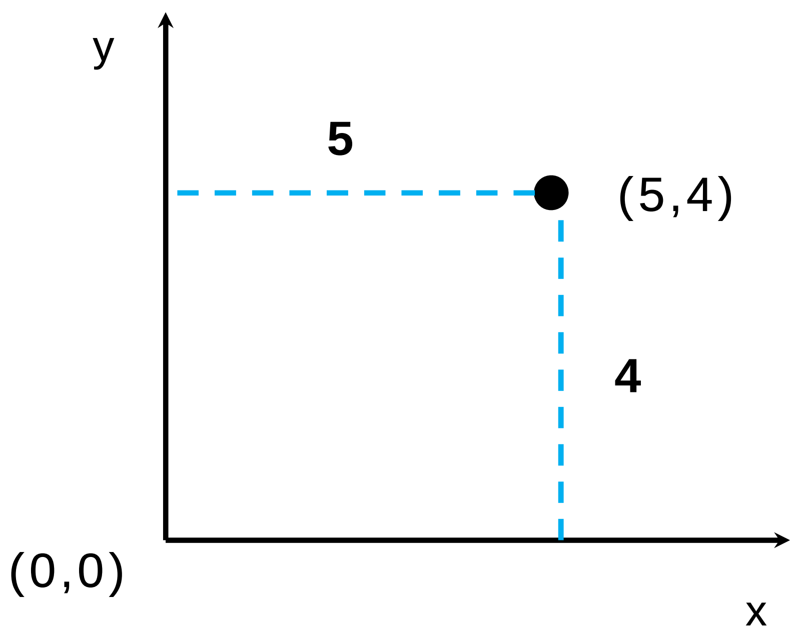
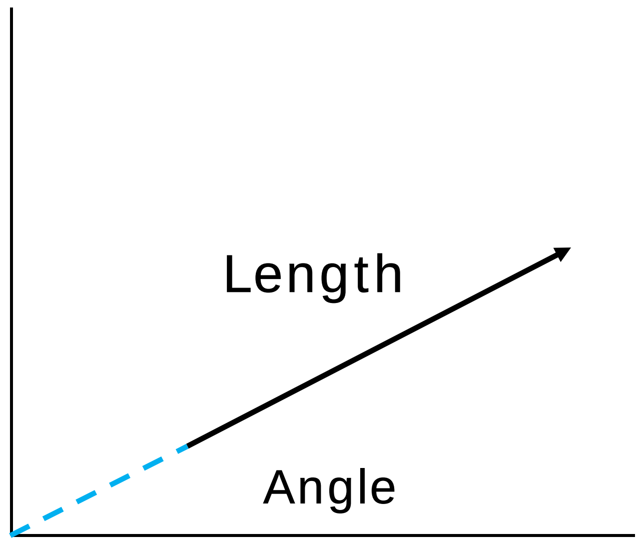
Points, Scalars and Vectors

- Points, vectors defined relative to a coordinate system
- **Point:** Location in coordinate system
- Example: Point (5,4)



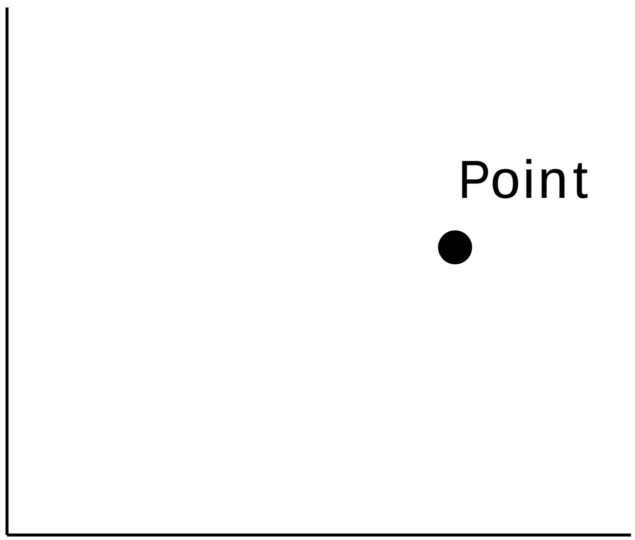
Vectors

- Magnitude
- Direction
- **NO** position
- Can be added, scaled, rotated
- CG vectors: 2, 3 or 4 dimensions



Points

- Cannot add or scale **points**
- Subtract **2 points** = **vector**



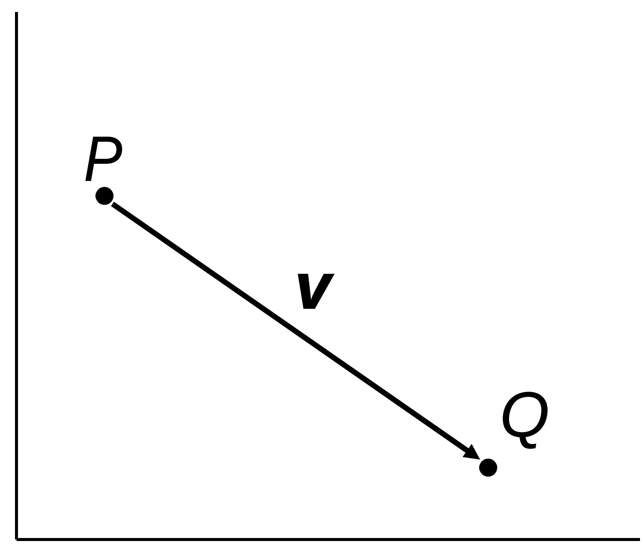
Vector-Point Relationship

- Diff. b/w 2 points = vector

$$\mathbf{v} = Q - P$$

- point + vector = point

$$P + \mathbf{v} = Q$$



Vector Operations

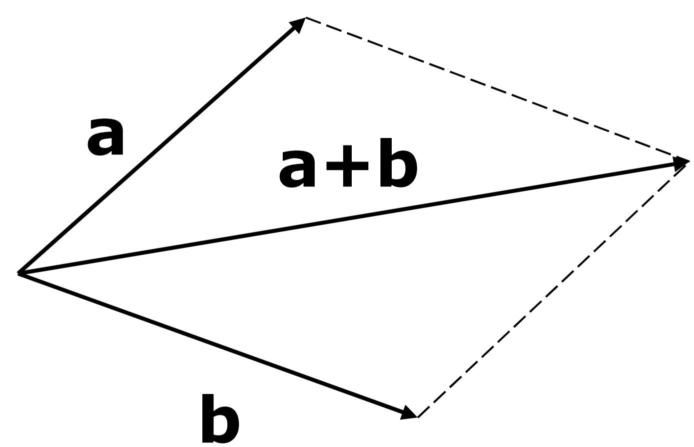
- Define vectors

$$\mathbf{a} = (a_1, a_2, a_3)$$

$$\mathbf{b} = (b_1, b_2, b_3)$$

Then vector addition:

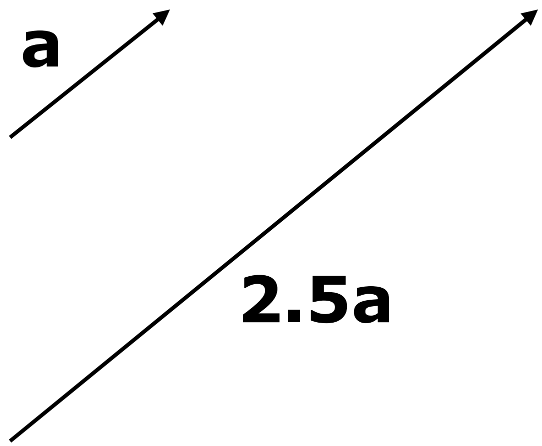
$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$



Vector Operations

- Define scalar, *s*
- Scaling vector by a scalar

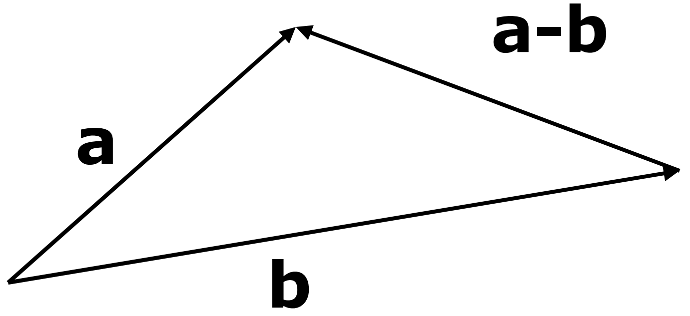
$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s)$$



Note vector subtraction:

$$\mathbf{a} - \mathbf{b}$$

$$= (a_1 + (-b_1), a_2 + (-b_2), a_3 + (-b_3))$$



Vector Operations: Examples

- Scaling vector by a scalar

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s)$$

- Vector addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_{1,} a_2 + b_2, a_3 + b_3)$$

- For example, if **a**=(2,5,6) and **b**=(-2,7,1) and *s*=6, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_{1,} a_2 + b_2, a_3 + b_3) = (0,12,7)$$

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s) = (12,30,36)$$

Affine Combination

- Given a vector

$$\mathbf{a} = (a_{1,} a_2, a_3, \ldots, a_n)$$

$$a_1 + a_2 + \ldots\ldots\ldots a_n = 1$$

- Affine combination: Sum of all components = 1

- Convex affine = affine + no negative component i.e

$$a_1, a_2, \ldots\ldots\ldots a_n = non - negative$$

Magnitude of a Vector

- Magnitude of **a**

$$|\mathbf{a}| = \sqrt{a_1{}^2 + a_2{}^2 .... + a_n{}^2}$$

- Normalizing a vector (unit vector)

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{vector}{magnitude}$$

- Note magnitude of normalized vector = 1. i.e

$$\sqrt{a_1{}^2 + a_2{}^2 .... + a_n{}^2} = 1$$

Magnitude of a Vector

- Example: if a = (2, 5, 6)

- Magnitude of **a**

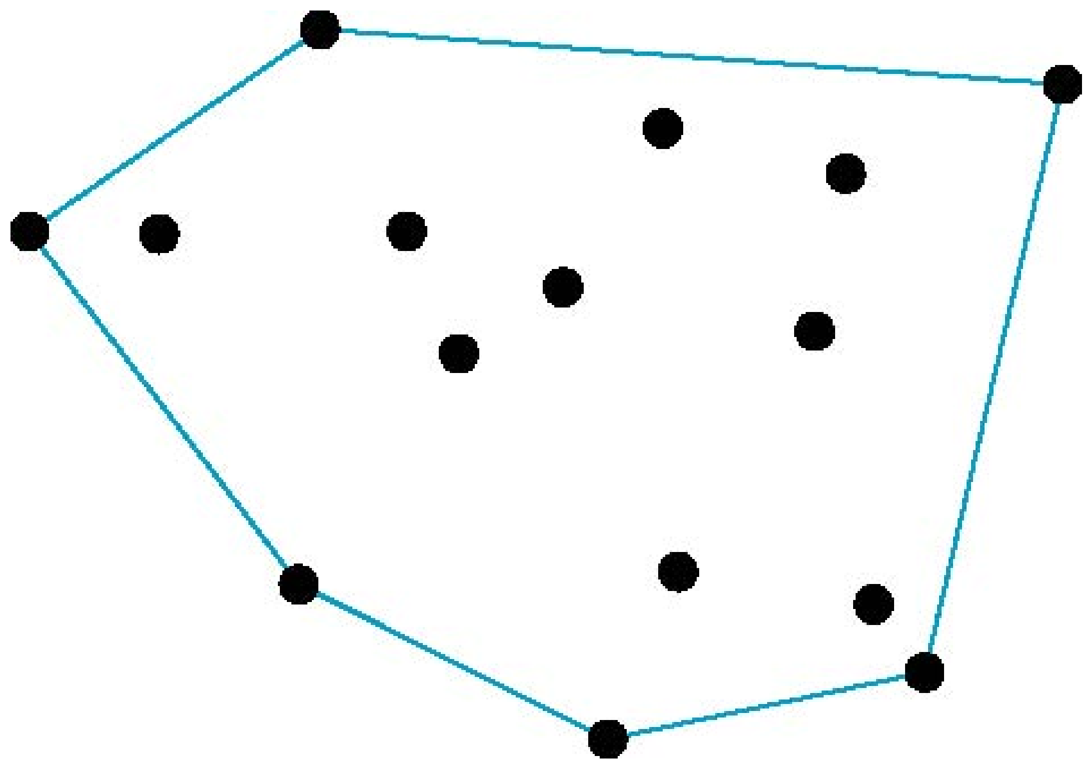
$$|\mathbf{a}| = \sqrt{2^2 + 5^2 + 6^2} = \sqrt{65}$$

- Normalizing **a**

$$\hat{\mathbf{a}} = \left(\frac{2}{\sqrt{65}}, \frac{5}{\sqrt{65}}, \frac{6}{\sqrt{65}} \right)$$

Convex Hull

- Smallest convex object containing $P_1, P_2, \ldots.. P_n$
- Formed by “shrink wrapping” points



Dot Product (Scalar product)

- Dot product,

$$d = \mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 \ldots\ldots\ldots + a_3 \cdot b_3$$

- For example, if *a*=(2,3,1) and *b*=(0,4,-1) then

$$a \cdot b = (2 \times 0) + (3 \times 4) + (1 \times -1)$$
$$= 0 + 12 - 1 = 11$$

Properties of Dot Products

- Symmetry (or commutative):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- Linearity:

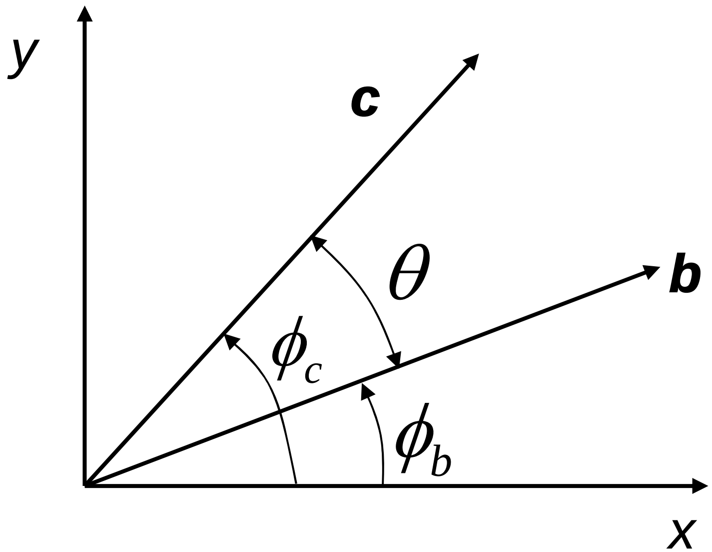
$$(\mathbf{a} + \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$$

- Homogeneity:

$$(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$$

- And $$\left|\mathbf{b}^2\right| = \mathbf{b} \cdot \mathbf{b}$$

Angle Between Two Vectors

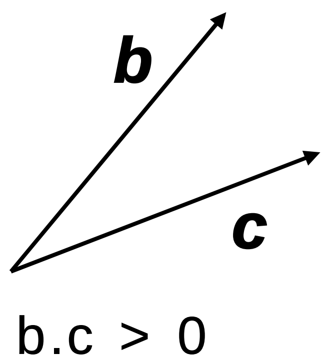


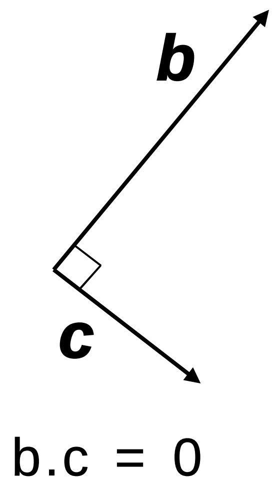
$$\mathbf{b} = \left(|\mathbf{b}|\cos\phi_b, |\mathbf{b}|\sin\phi_b\right)$$

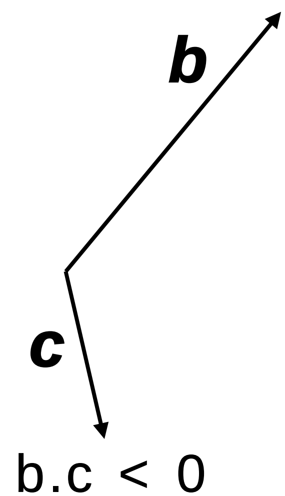
$$\mathbf{c} = \left(|\mathbf{c}|\cos\phi_c, |\mathbf{c}|\sin\phi_c\right)$$

$$\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}||\mathbf{c}|\cos\theta$$

Sign of **b.c**:







Angle Between Two Vectors

- Find the angle b/w the vectors **b** = (3,4) and **c =** (5,2)

Angle Between Two Vectors

- Find the angle b/w vectors **b** = (3,4) and **c = (5,2)**
- **Step 1:** Find magnitudes of vectors **b** and **c**

$$|\mathbf{b}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|\mathbf{c}| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

- **Step 2:** Normalize vectors **b** and **c**

$$\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad \hat{\mathrm{c}} = \left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right)$$

Angle Between Two Vectors

- **Step 3:** Find angle as dot product $\hat{\mathbf{b}} \bullet \hat{\mathbf{c}}$

$$\hat{\mathbf{b}} \bullet \hat{\mathbf{c}} = \left(\frac{3}{5}, \frac{4}{5}\right) \bullet \left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right)$$

$$\hat{\mathbf{b}} \bullet \hat{\mathbf{c}} = \frac{15}{5\sqrt{29}} + \frac{8}{5\sqrt{29}} = \frac{23}{5\sqrt{29}} = 0.85422$$

- **Step 4:** Find angle as inverse cosine

$$\theta = \cos(0.85422) = 31.326^{\circ}$$

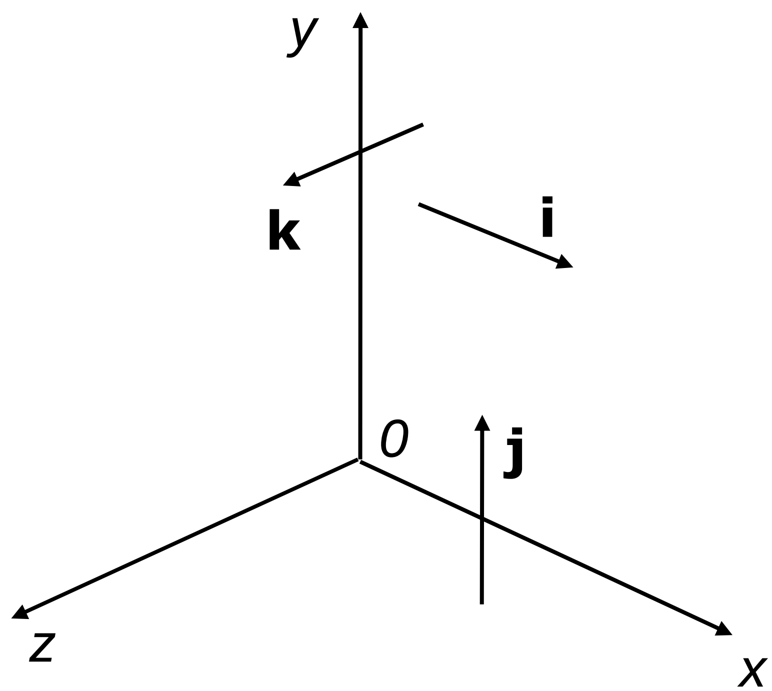
Standard Unit Vectors

Define

$$\mathbf{i} = (1,0,0)$$

$$\mathbf{j} = (0,1,0)$$

$$\mathbf{k} = (0,0,1)$$



So that any vector,

$$\mathbf{v} = (a,b,c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Cross Product (Vector product)

If

$$\mathbf{a} = (a_x, a_y, a_z) \qquad \mathbf{b} = (b_x, b_y, b_z)$$

Then

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} - (a_x b_z - a_z b_x)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

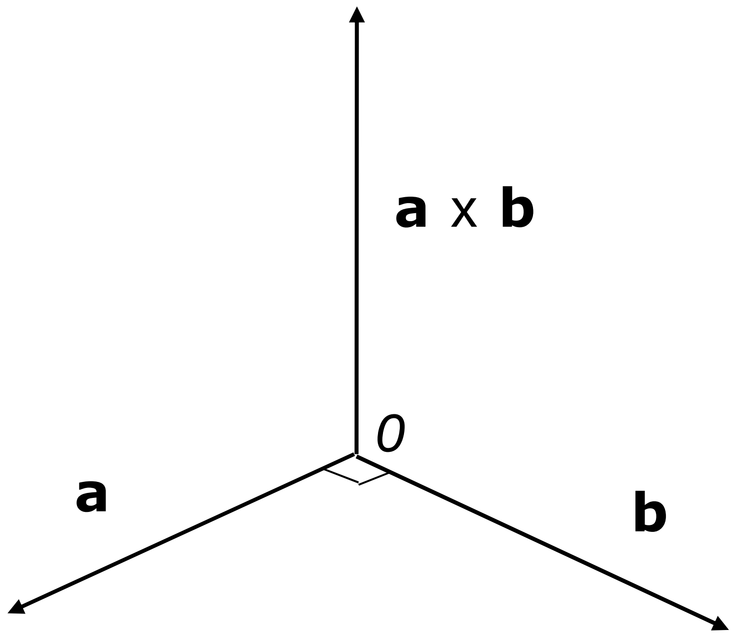
Remember using determinant

$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Note: a x **b** is perpendicular to **a** and **b**

Cross Product

Note: **a** x **b** is perpendicular to both **a** and **b**



Cross Product

Calculate **a x b** if a = (3,0,2) and **b** = (4,1,8)

Cross Product (Vector product)

Calculate **a x b** if a = (3,0,2) and **b** = (4,1,8)

$$\mathbf{a} = (3,0,2) \qquad \mathbf{b} = (4,1,8)$$

Using determinant

$$\begin{vmatrix} i & j & k \\ 3 & 0 & 2 \\ 4 & 1 & 8 \end{vmatrix}$$

Then

$$\mathbf{a} \times \mathbf{b} = (0-2)\mathbf{i} - (24-8)\mathbf{j} + (3-0)\mathbf{k}$$

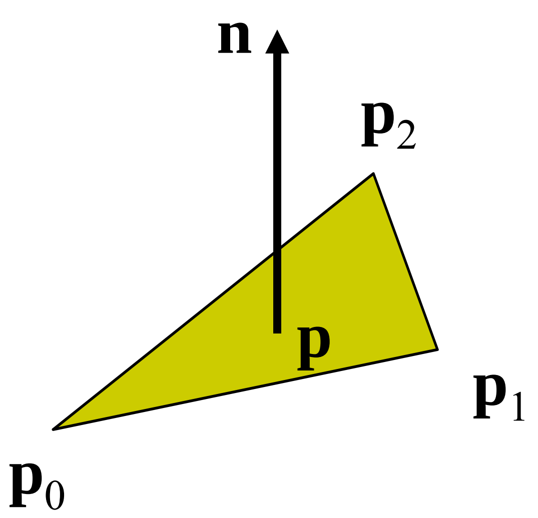
$$= -2\mathbf{i} - 16\mathbf{j} + 3\mathbf{k}$$

Normal for Triangle using Cross Product Method

plane $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$

$\mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_0) \times (\mathbf{p}_1 - \mathbf{p}_0)$

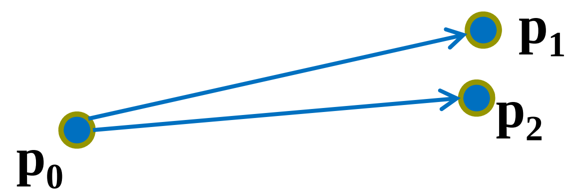
normalize $\mathbf{n} \leftarrow \mathbf{n} / |\mathbf{n}|$



Note that right-hand rule determines outward face

Newell Method for Normal Vectors

- Problems with cross product method:
  - calculation difficult by hand, tedious
  - If 2 vectors almost parallel, cross product is small
  - Numerical inaccuracy may result



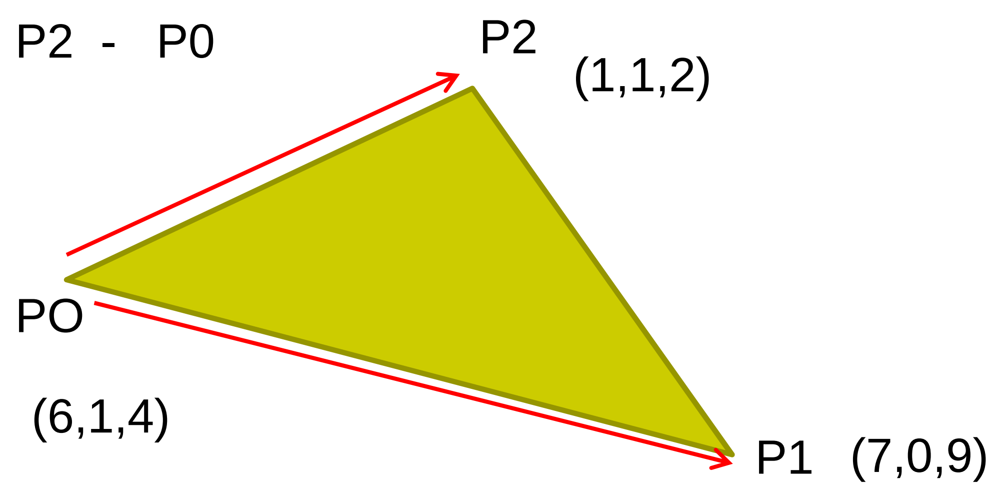
- Proposed by Martin Newell at Utah (teapot guy)
  - Uses formulae, suitable for computer
  - Compute during mesh generation
  - Robust!

Newell Method Example

- Example: Find normal of polygon with vertices
  P0 = (6,1,4), P1=(7,0,9) and P2 = (1,1,2)

- Using simple cross product:
  ((7,0,9)-(6,1,4)) X ((1,1,2)-(6,1,4)) = (2,-23,-5)

P1 - P0



Newell Method for Normal Vectors

- Formulae: Normal N = (mx, my, mz)

$$m_x = \sum_{i=0}^{N-1} \left(y_i - y_{next(i)}\right)\left(z_i + z_{next(i)}\right)$$

$$m_y = \sum_{i=0}^{N-1} \left(z_i - z_{next(i)}\right)\left(x_i + x_{next(i)}\right)$$

$$m_z = \sum_{i=0}^{N-1} \left(x_i - x_{next(i)}\right)\left(y_i + y_{next(i)}\right)$$

Newell Method for Normal Vectors

- Calculate x component of normal

$$m_x = \sum_{i=0}^{N-1} \left(y_i - y_{next(i)}\right)\left(z_i + z_{next(i)}\right)$$

$$m_x = (1)(13) + (-1)(11) + (0)(6)$$

$$m_x = 13 - 11 + 0$$

$$m_x = 2$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4

Newell Method for Normal Vectors

- Calculate y component of normal

$$m_y = \sum_{i=0}^{N-1} \left(z_i - z_{next(i)}\right)\left(x_i + x_{next(i)}\right)$$

$$m_y = (-5)(13) + (7)(8) + (-2)(7)$$

$$m_y = -65 + 56 - 14$$

$$m_y = -23$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4

Newell Method for Normal Vectors

- Calculate z component of normal

$$m_z = \sum_{i=0}^{N-1} \left(x_i - x_{next(i)}\right)\left(y_i + y_{next(i)}\right)$$

$$m_z = (-1)(1) + (6)(1) + (-5)(2)$$

$$m_z = -1 + 6 - 10$$

$$m_z = -5$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4

Note: Using Newell method yields same result as Cross product method (2,-23,-5)

Finding Vector Reflected From a Surface

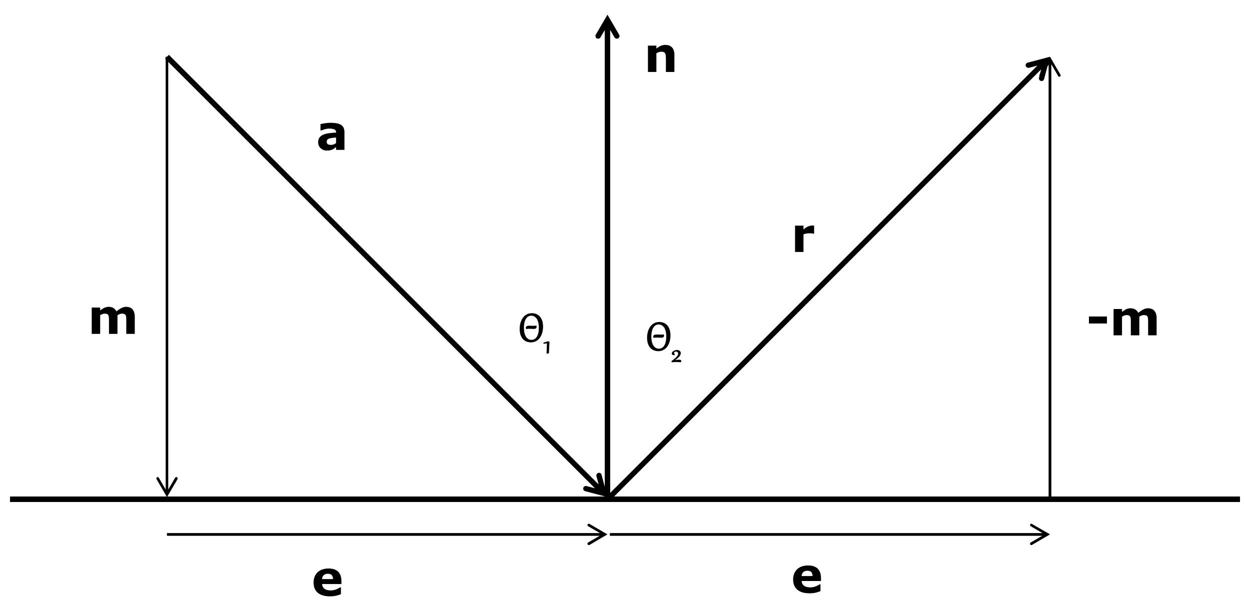
- **a** = original vector
- **n** = normal vector
- **r** = reflected vector
- **m** = projection of **a** along **n**
- **e** = projection of **a** orthogonal to **n**

Note: $\Theta_1 = \Theta_2$

$$\mathbf{e} = \mathbf{a} - \mathbf{m}$$

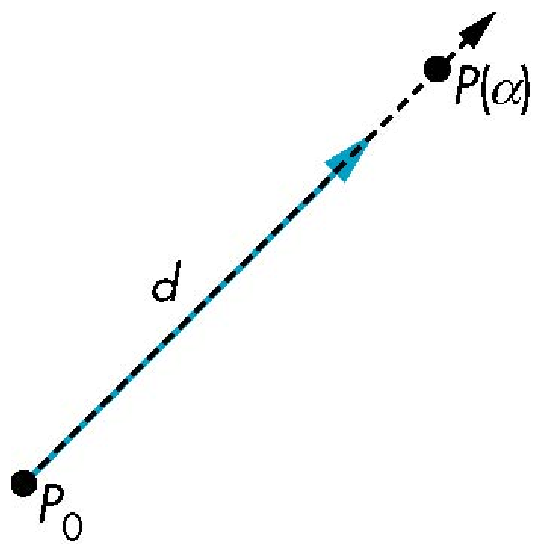
$$\mathbf{r} = \mathbf{e} - \mathbf{m}$$

$$\Rightarrow \mathbf{r} = \mathbf{a} - 2\mathbf{m}$$



Lines

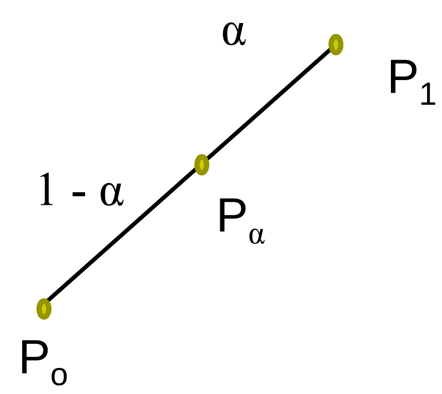
- Consider all points of the form
  - $P(\alpha)=P_0 + \alpha \mathbf{d}$
  - **Line:** Set of all points that pass through P_0 in direction of vector $\mathbf{d}$



Parametric Form

- Two-dimensional forms of a line
  - **Explicit:** $y = mx + h$
  - **Implicit:** $ax + by + c = 0$
  - **Parametric:**

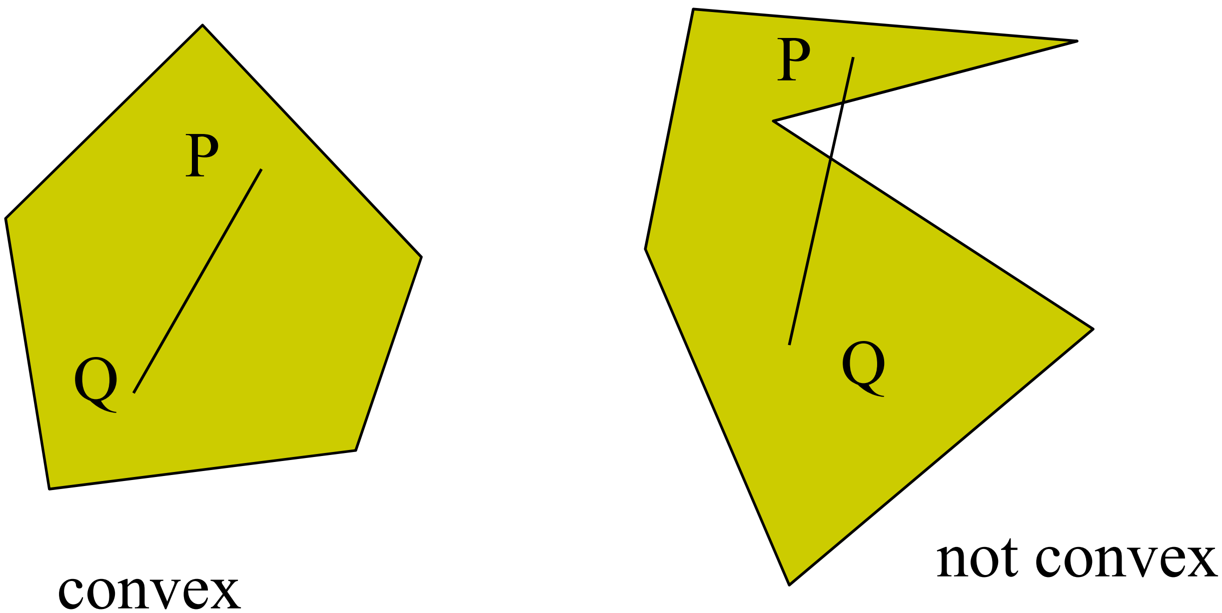
$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$
$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$



- Parametric form of line
  - More robust and general than other forms
  - Extends to curves and surfaces

Convexity

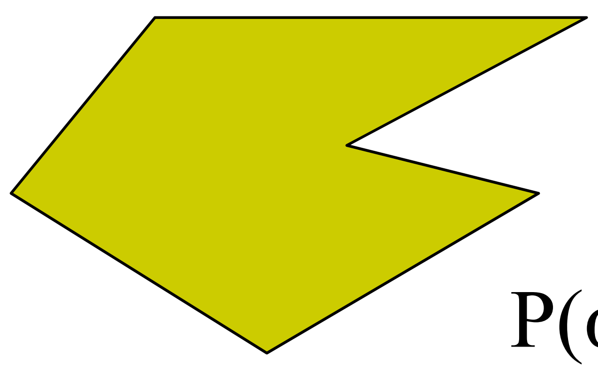
- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



Curves and Surfaces

- **Curves:** 1-parameter **non-linear** functions of the form $P(\alpha)$
- **Surfaces:** two-parameter functions $P(\alpha, \beta)$
  - Linear functions give planes and polygons

$P(\alpha)$



$P(\alpha, \beta)$

References

- Angel and Shreiner, Interactive Computer Graphics, 6th edition, Chapter 3
- Hill and Kelley, Computer Graphics using OpenGL, 3rd edition, Sections 4.2 - 4.4