Computer Graphics
CS 4731 Lecture 24
Curves

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So Far…

- Dealt with straight lines and flat surfaces
- Real world objects include curves
- Need to develop:
  - Representations of curves (mathematical)
  - Tools to render curves

Interactive Curve Design

- Mathematical formula unsuitable for designers
- Prefer to interactively give sequence of points (control points)
- Write procedure:
  - **Input:** sequence of points
  - **Output:** parametric representation of curve

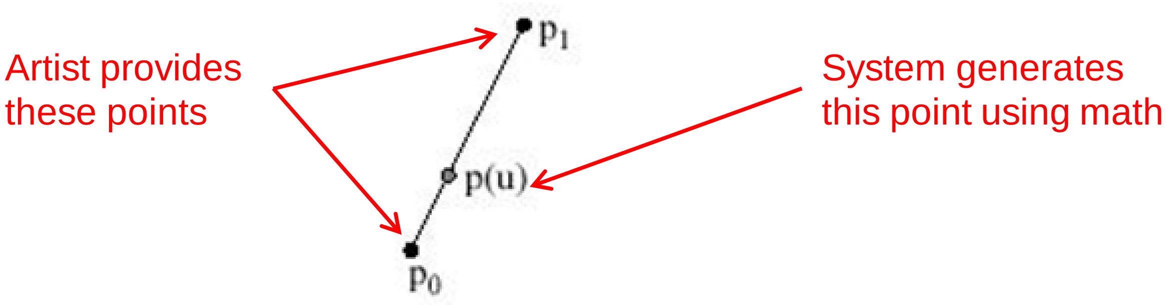
Interactive Curve Design

- 1 approach: curves pass through control points (interpolate)
- **Example:** Lagrangian Interpolating Polynomial
- Difficulty with this approach:
    - Polynomials always have "wiggles"
    - For straight lines wiggling is a problem
- Our approach: approximate control points (Bezier, B-Splines)

De Casteljau Algorithm

- Consider smooth curve that approximates sequence of control points [p0,p1,....]

$$p(u) = (1-u)p_0 + up_1 \qquad 0 \le u \le 1$$

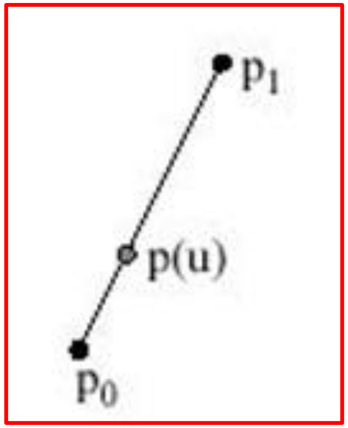


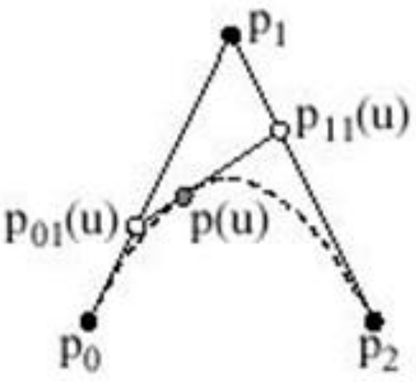
- Blending functions: *u* and *(1 – u)* are non-negative and sum to one

De Casteljau Algorithm

- Now consider 3 points
- 2 line segments, P0 to P1 and P1 to P2

$$p_{01}(u) = (1-u)\,p_0 + u p_1 \qquad p_{11}(u) = (1-u)\,p_1 + u p_2$$





De Casteljau Algorithm

Substituting known values of $p_{01}(u)$ and $p_{11}(u)$

$$p(u) = (1-u)\,p_{01} + u p_{11}(u)$$

$$= \underbrace{(1-u)^2}_{b_{02}(u)} \boxed{p_0} + \underbrace{(2u(1-u))}_{b_{12}(u)} \boxed{p_1} + \underbrace{u^2}_{b_{22}(u)} \boxed{p_2}$$

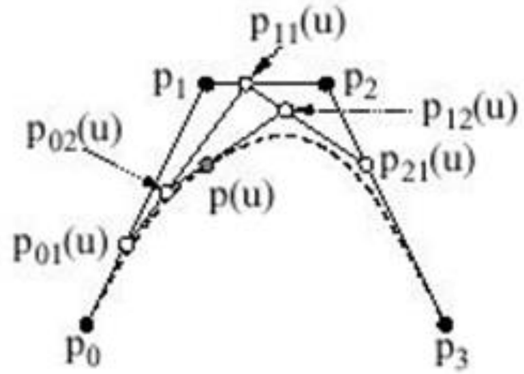
Blending functions for degree 2 Bezier curve

$$b_{02}(u) = (1-u)^2 \qquad b_{12}(u) = 2u(1-u) \qquad b_{22}(u) = u^2$$

Note: blending functions, non-negative, sum to 1

De Casteljau Algorithm

- Extend to 4 control points P0, P1, P2, P3



$$p(u) = \underbrace{(1-u)^3}_{b_{03}(u)} \boxed{p_0} + \underbrace{(3u(1-u)^2)}_{b_{13}(u)} \boxed{p_1} + \underbrace{(3u^2(1-u))}_{b_{23}(u)} \boxed{p_2} + \underbrace{u^3}_{b_{33}(u)}$$

- Final result above is Bezier curve of degree 3

De Casteljau Algorithm

$$p(u) = (1-u)^3 \boxed{p_0} + (3u(1-u)^2) \boxed{p_1} + (3u^2(1-u)) \boxed{p_2} + u^3$$

$b_{03}(u)$ $\quad$ $b_{13}(u)$ $\quad$ $b_{23}(u)$ $\quad$ $b_{33}(u)$

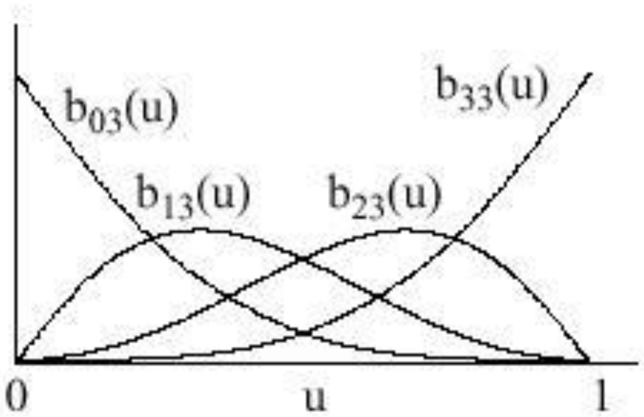
- Blending functions are polynomial functions called **Bernstein's polynomials**

$$b_{03}(u) = (1-u)^3$$

$$b_{13}(u) = 3u(1-u)^2$$

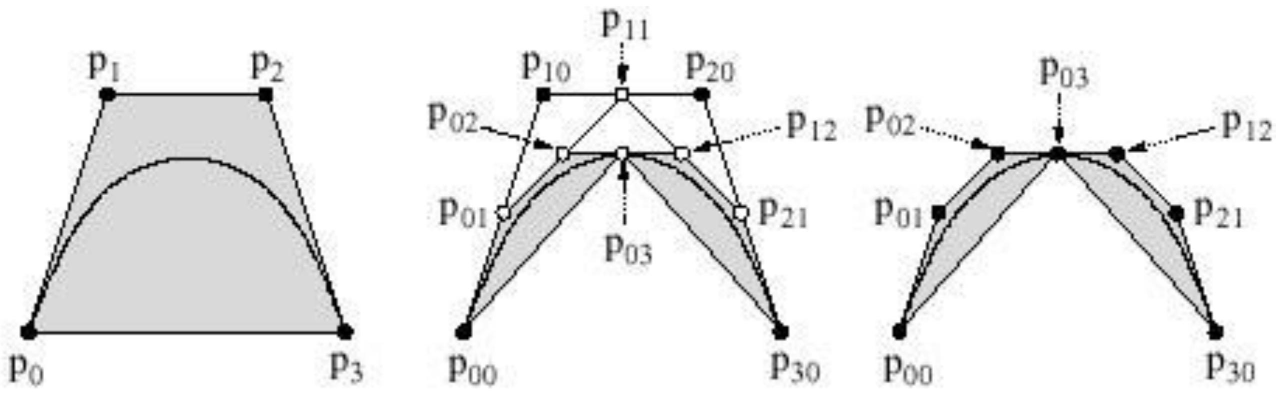
$$b_{23}(u) = 3u^2(1-u)$$

$$b_{33}(u) = u^3$$



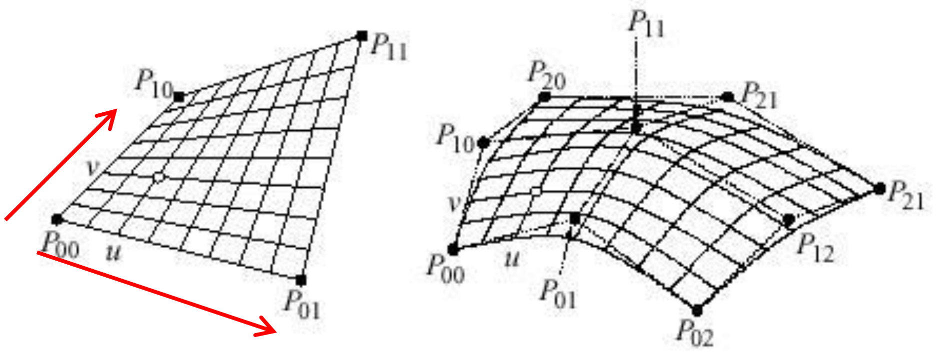
Subdividing Bezier Curves

- OpenGL renders flat objects
- To render curves, approximate with small linear segments
- Subdivide surface to polygonal patches
- Bezier Curves can either be straightened or curved recursively in this way



Bezier Surfaces

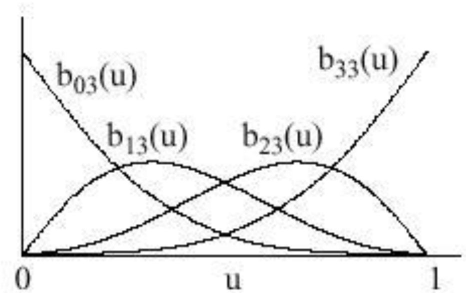
- Bezier surfaces: interpolate in two dimensions
- This called Bilinear interpolation
- Example: 4 control points, P00, P01, P10, P11,
  - 2 parameters **u** and **v**
- Interpolate between
  - P00 and P01 using u
  - P10 and P11 using u
  - P00 and P10 using v
  - P01 and P11 using v



$$p(u,v) = (1-v)((1-u)\,p_{00} + up_{01}) + v((1-u)\,p_{10} + up_{11})$$

Problems with Bezier Curves

- Bezier curves elegant but to achieve smoother curve
  - = more control points
  - = higher order polynomial
  - = more calculations

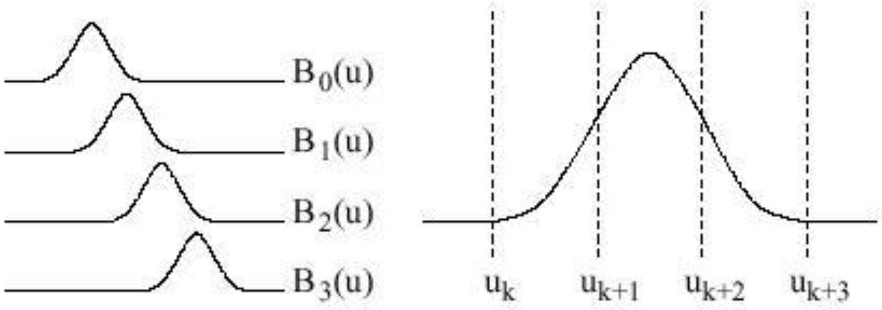


- **Global support problem:** All blending functions are non-zero for all values of *u*
- All control points contribute to all parts of the curve
- Means after modelling complex surface (e.g. a ship), if one control point is moves, recalculate everything!

B-Splines

- B-splines designed to address Bezier shortcomings
- B-Spline given by blending control points
- **Local support:** Each spline contributes in limited range
- Only non-zero splines contribute in a given range of *u*

$$p(u) = \sum_{i=0}^{m} B_i(u)\, p_i$$



B-spline blending functions, order 2

NURBS

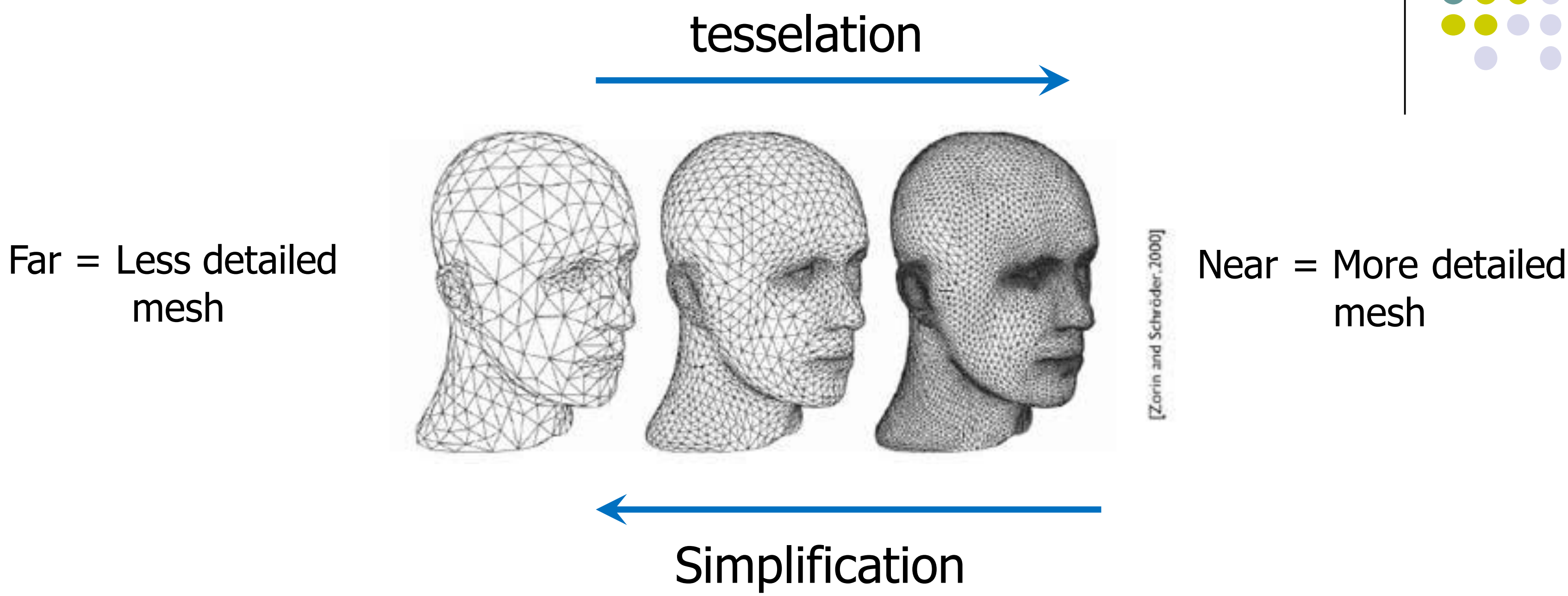
- Non-uniform Rational B-splines (NURBS)
- Rational function means ratio of two polynomials
- Some curves can be expressed as rational functions but not as simple polynomials
- No known exact polynomial for circle
- Rational parametrization of unit circle on xy-plane:

$$x(u) = \frac{1-u^2}{1+u^2}$$

$$y(u) = \frac{2u}{1+u^2}$$

$$z(u) = 0$$

Tesselation

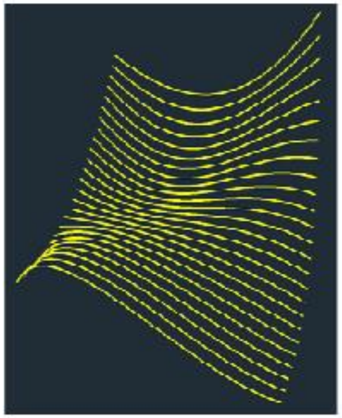


- **Previously:** Pre-generate mesh versions offline
- Tesselation shader unit new to GPU in DirectX 10 (2007)
  - Subdivide faces **on-the-fly** to yield finer detail, generate new vertices, primitives
- Mesh simplification/tesselation on GPU = Real time LoD

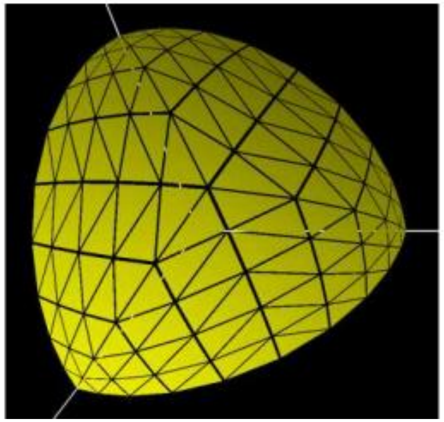
Tessellation Shaders

- Can subdivide curves, surfaces on the GPU

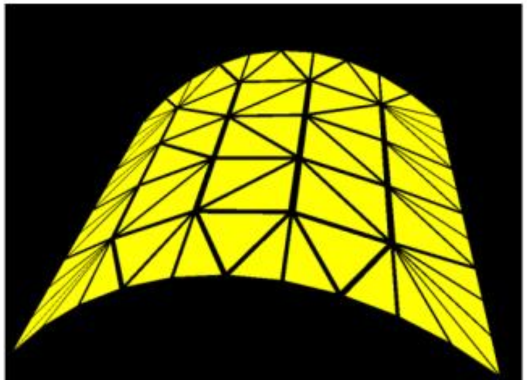
Lines



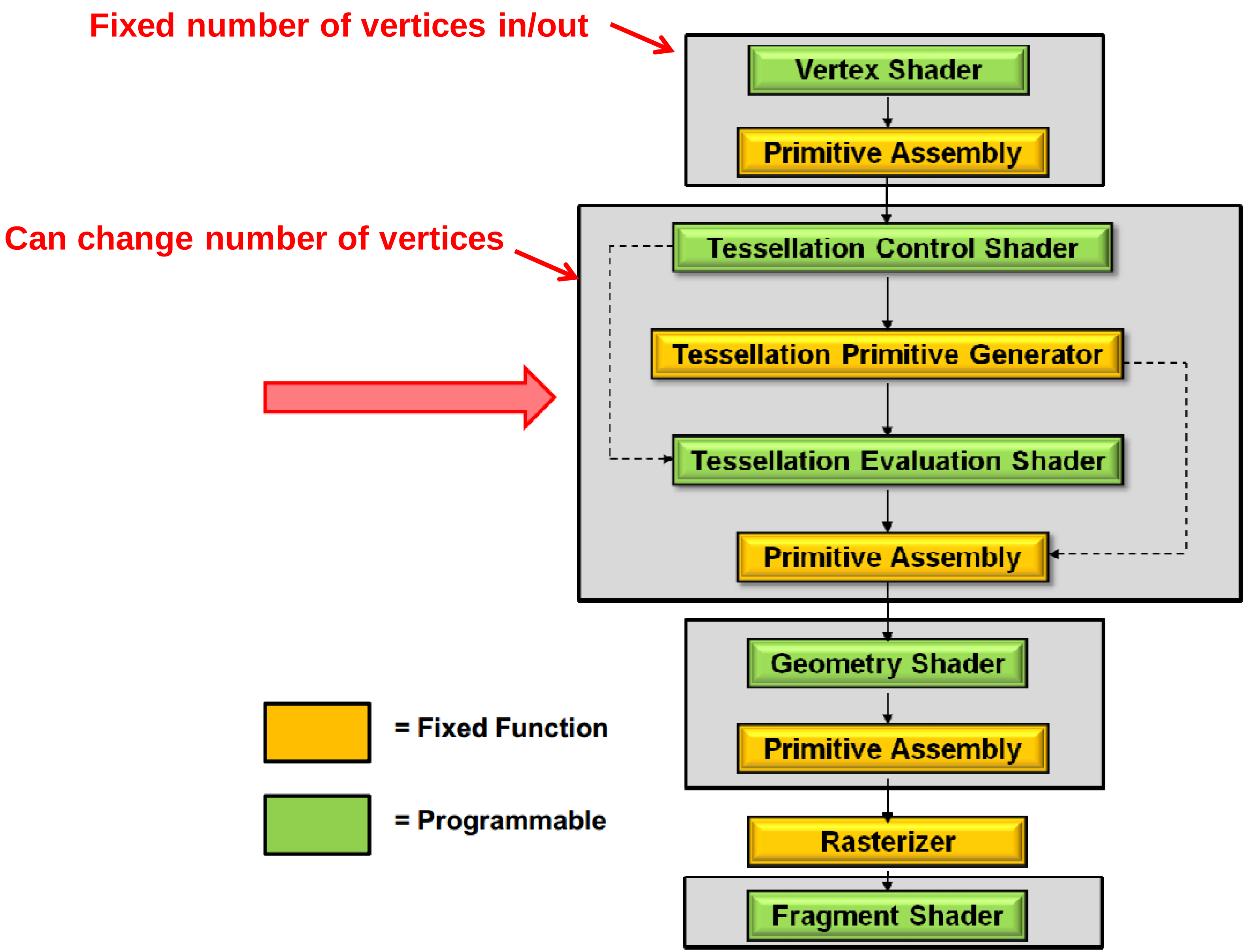
Triangles



Quads (subsequently broken into triangles)

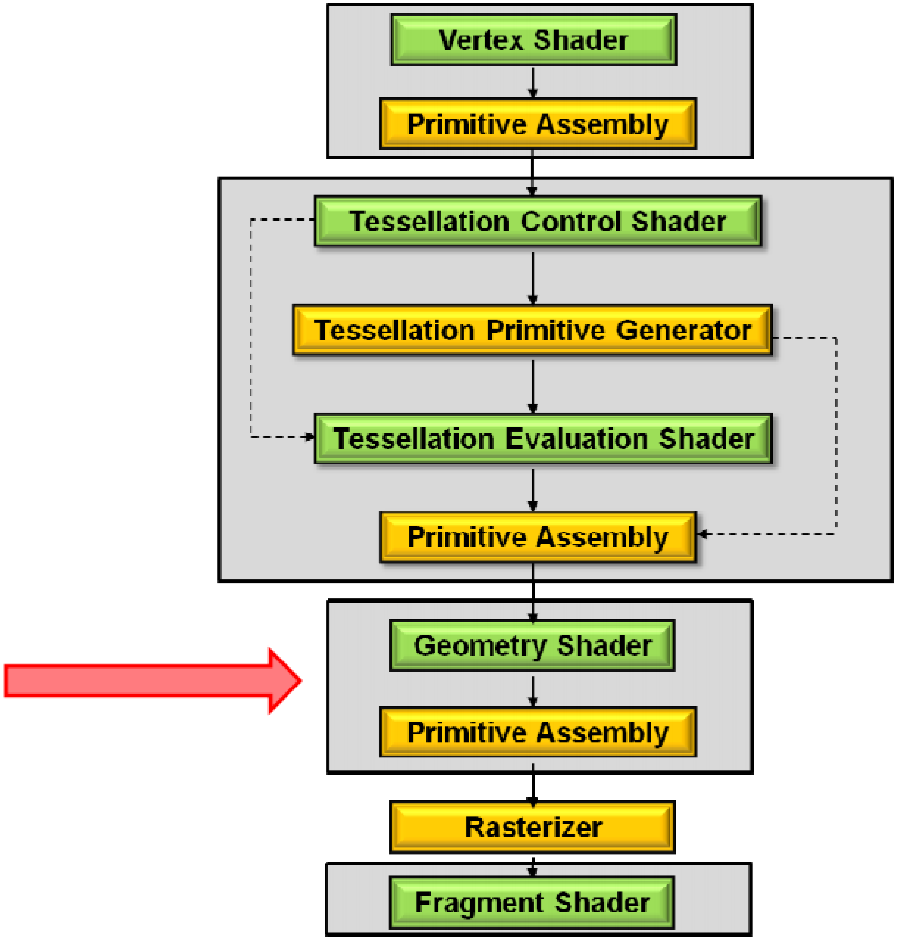


Where Does Tesselation Shader Fit?



Geometry Shader

- After Tesselation shader. Can
  - Handle whole primitives
  - Generate new primitives
  - Generate no primitives (cull)



References

- Hill and Kelley, chapter 11
- Angel and Shreiner, Interactive Computer Graphics, 6th edition, Chapter 10
- Shreiner, OpenGL Programming Guide, 8th edition