Computer Graphics (CS 4731)
Lecture 14: Projection (Part 2): Derivation

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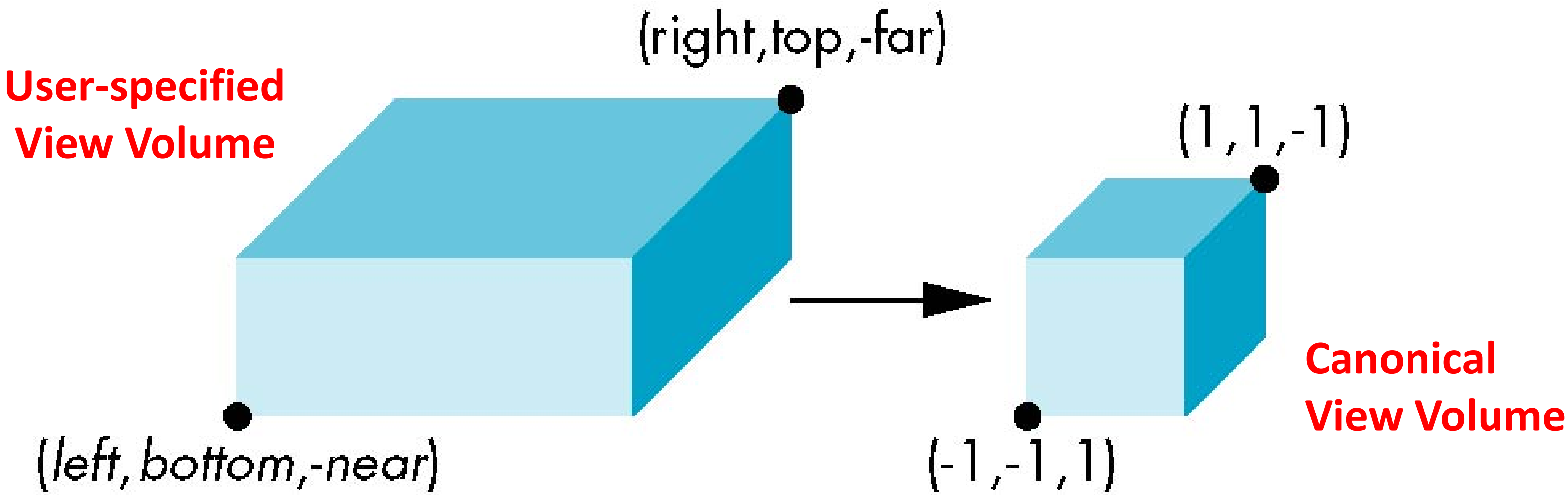
Parallel Projection

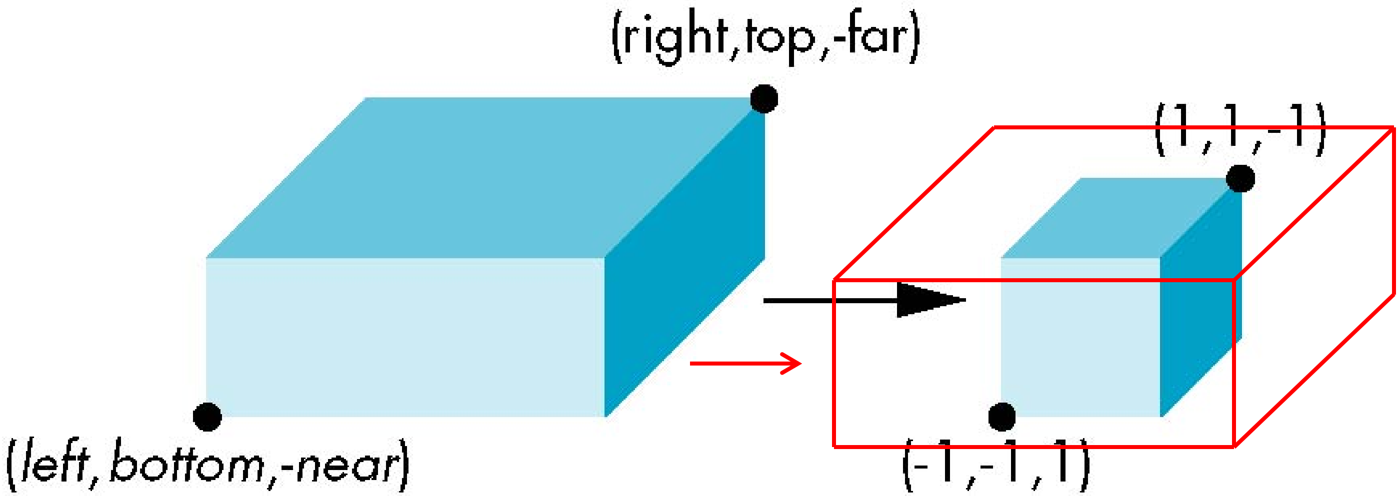
- **normalization** $\Rightarrow$ find 4x4 matrix to transform **user-specified view volume** to **canonical view volume (cube)**



glOrtho(left, right, bottom, top,near, far)

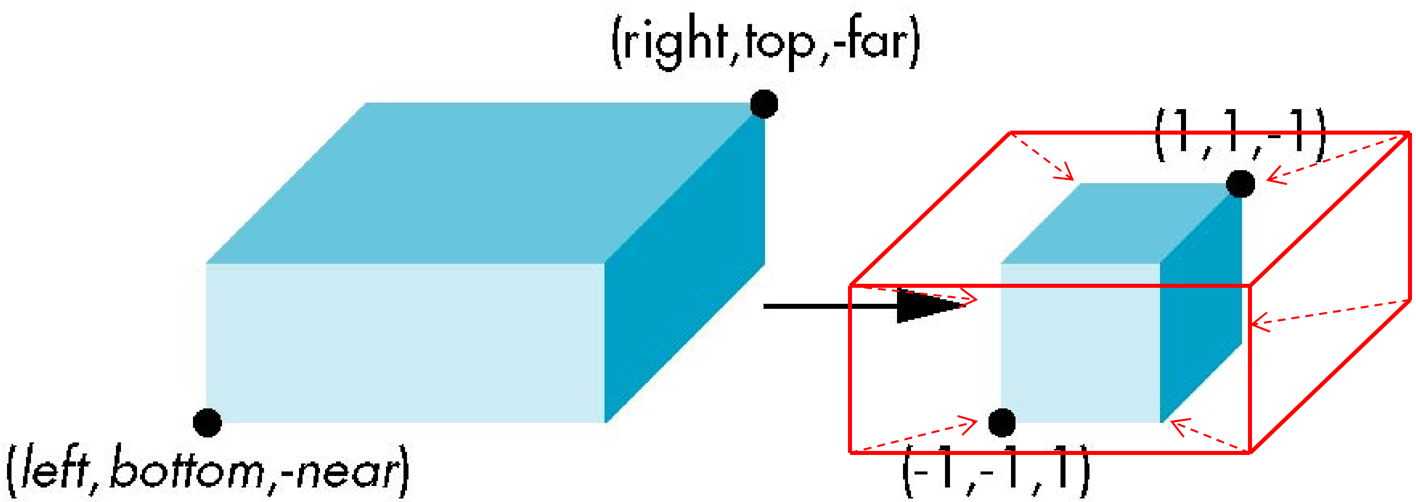
Parallel Projection: Ortho

- Parallel projection: 2 parts
  1. **Translation:** centers view volume at origin



Parallel Projection: Ortho

2. **Scaling:** reduces user-selected cuboid to canonical cube (dimension 2, centered at origin)



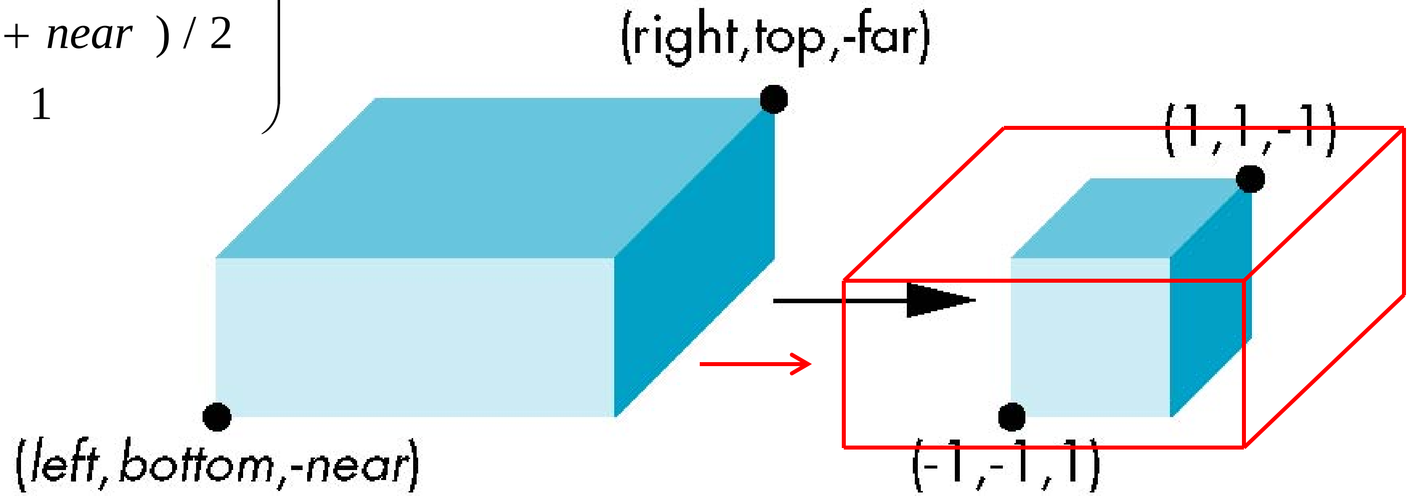
Parallel Projection: Ortho

- Translation lines up midpoints: E.g. midpoint of x = (right + left)/2
- Thus translation factors:

  -(right + left)/2, -(top + bottom)/2, -(far+near)/2

- Translation matrix:

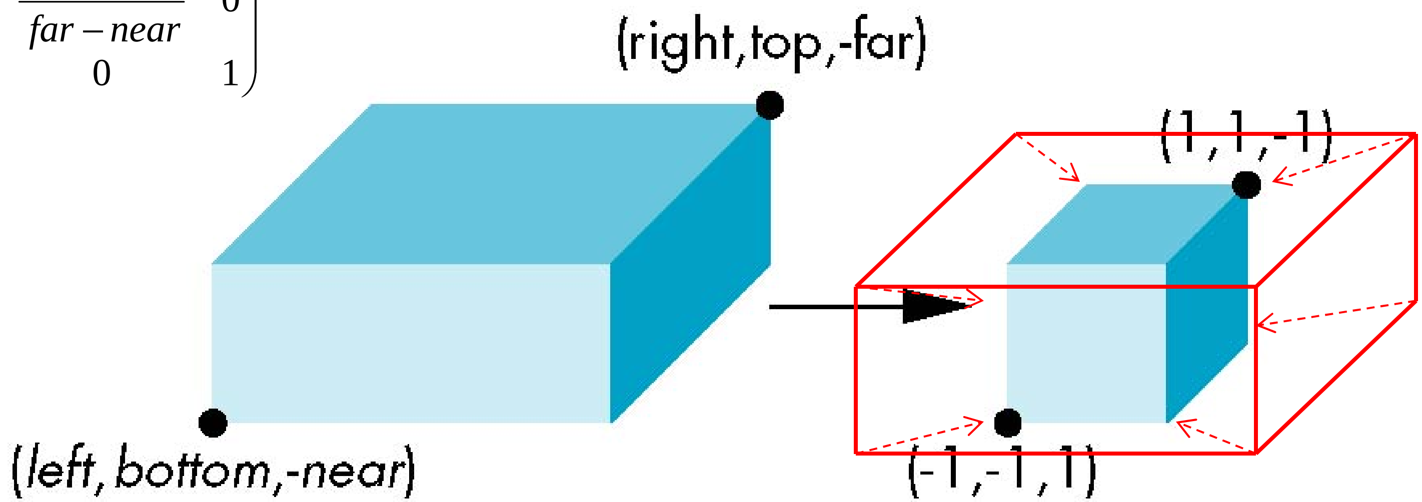
$$\begin{pmatrix} 1 & 0 & 0 & -(right + left)/2 \\ 0 & 1 & 0 & -(top + bottom)/2 \\ 0 & 0 & 1 & -(far + near)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Parallel Projection: Ortho

- Scaling factor: ratio of ortho view volume to cube dimensions
- Scaling factors: 2/(right - left), 2/(top - bottom), 2/(far - near)
- Scaling Matrix M2:

$$\begin{pmatrix} \frac{2}{right-left} & 0 & 0 & 0 \\ 0 & \frac{2}{top-bottom} & 0 & 0 \\ 0 & 0 & \frac{2}{far-near} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Parallel Projection: Ortho

Concatenating **Translation** x **Scaling**, we get Ortho Projection matrix

$$
\begin{pmatrix}
\frac{2}{right - left} & 0 & 0 & 0 \\
0 & \frac{2}{top - bottom} & 0 & 0 \\
0 & 0 & \frac{2}{far - near} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 & 0 & -(right + left)/2 \\
0 & 1 & 0 & -(top + bottom)/2 \\
0 & 0 & 1 & -(far + near)/2 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
\mathbf{P} = \mathbf{ST} =
\begin{bmatrix}
\frac{2}{right - left} & 0 & 0 & -\frac{right - left}{right - left} \\
0 & \frac{2}{top - bottom} & 0 & -\frac{top + bottom}{top - bottom} \\
0 & 0 & \frac{2}{near - far} & \frac{far + near}{far - near} \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Final Ortho Projection

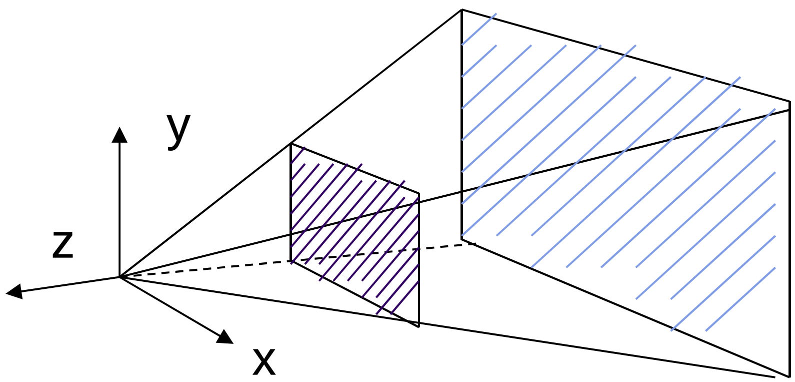
- Set $z = 0$
- Equivalent to the homogeneous coordinate transformation

$$\mathbf{M}_{\text{orth}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Hence, general orthogonal projection in 4D is $\mathbf{P} = \mathbf{M}_{\text{orth}}\mathbf{ST}$

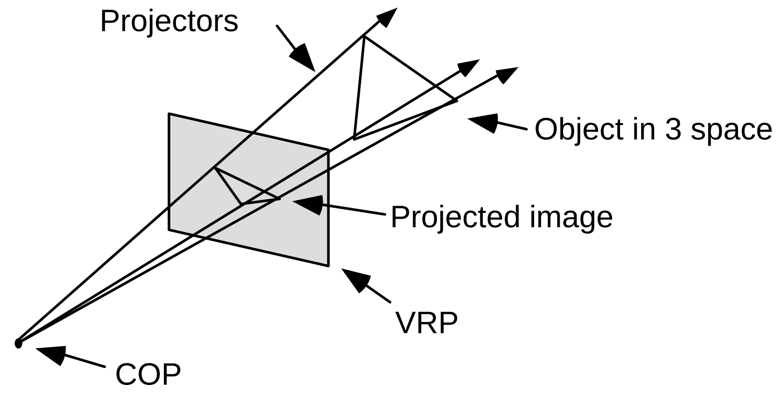
Perspective Projection

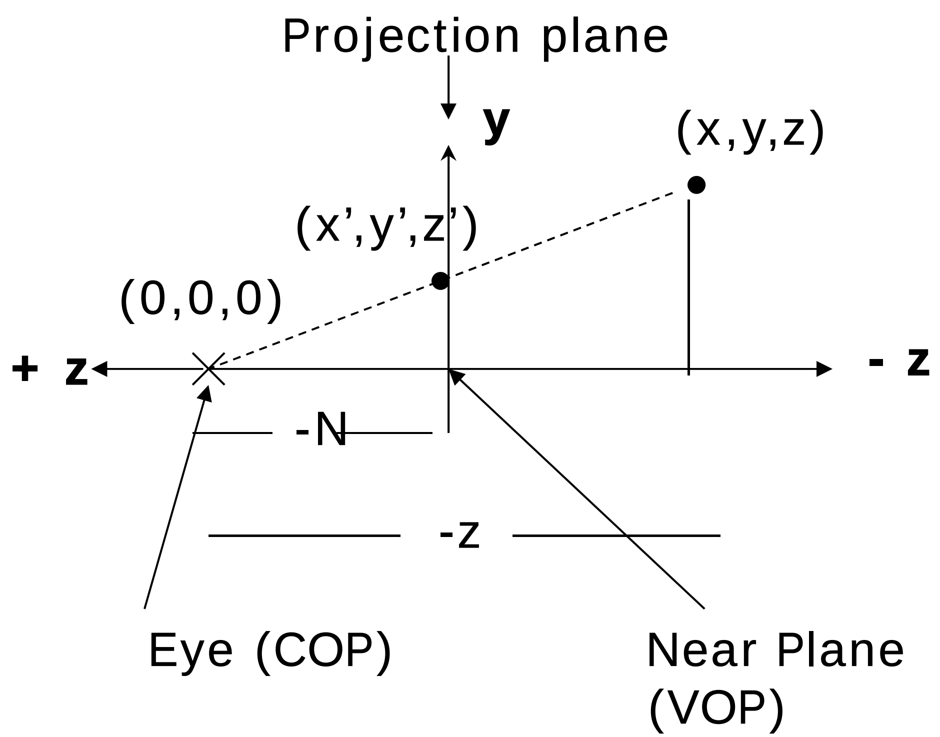
- Projection – map the object from 3D space to 2D screen



Perspective()
Frustrum( )

Perspective Projection: Classical





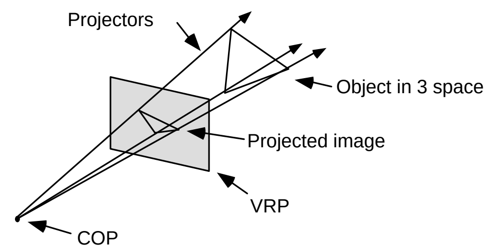
Based on similar triangles:

$$\frac{y'}{y} = \frac{N}{-z}$$

$$\Rightarrow \quad y' = y \times \frac{N}{-z}$$

Perspective Projection: Classical

- So (x*,y*) projection of point, (x,y,z) unto near plane N is given as:



$$(x^*, y^*) = \left(x\frac{N}{-z}, y\frac{N}{-z} \right)$$

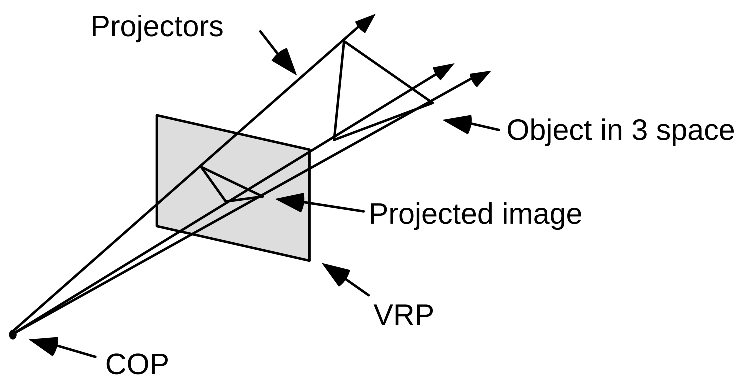
- Numerical example:

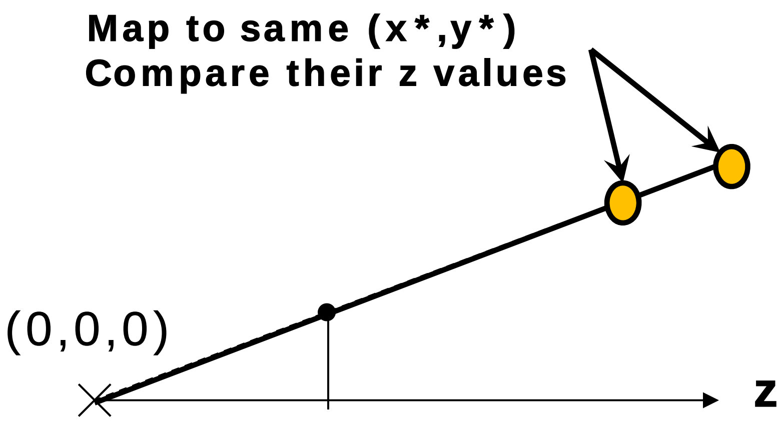
Q. Where on the viewplane does P = (1, 0.5, -1.5) lie for a near plane at N = 1?

$$(x^*, y^*) = \left(x\frac{N}{-z}, y\frac{N}{-z} \right) = \left(1\times\frac{1}{1.5}, 0.5\times\frac{1}{1.5} \right) = (0.666, 0.333)$$

Pseudodepth

- Classical perspective projection projects (x,y) coordinates to (x*, y*), drops z coordinates

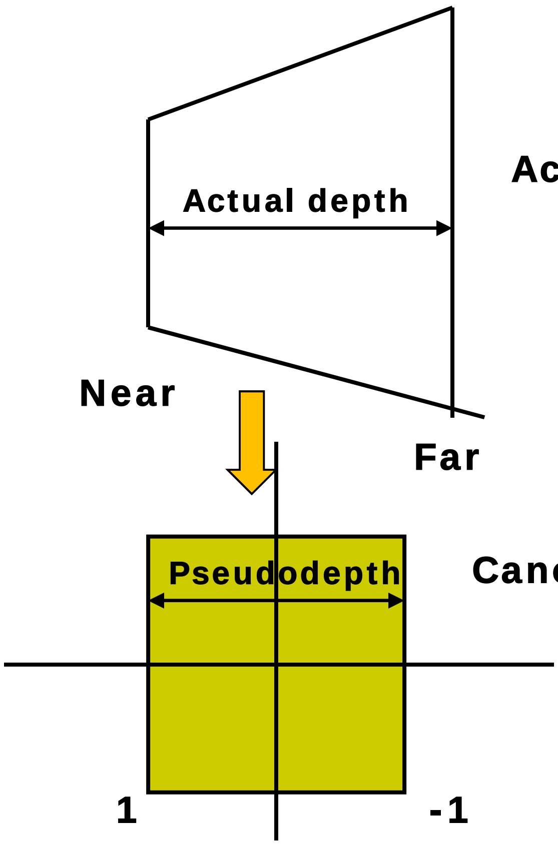




- But we need z to find closest object (depth testing)!!!

Perspective Transformation

- **Perspective transformation** maps actual z distance of perspective view volume to range [–1 to 1] (**Pseudodepth**) for canonical view volume



Actual view volume

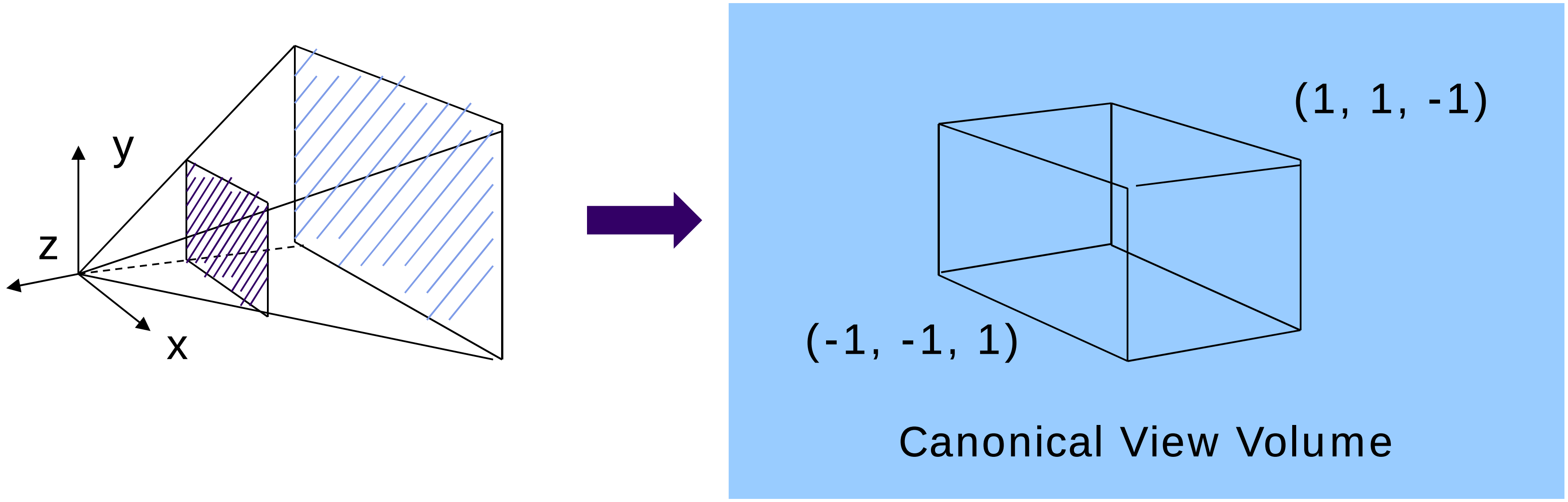
Canonical view volume

We want perspective Transformation and NOT classical projection!!

Set scaling z
Pseudodepth = az + b
Next solve for a and b

Perspective Transformation

- We want to transform viewing frustum volume into canonical view volume

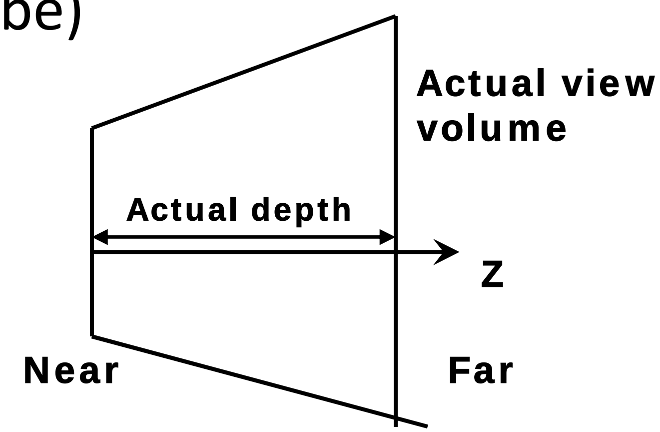


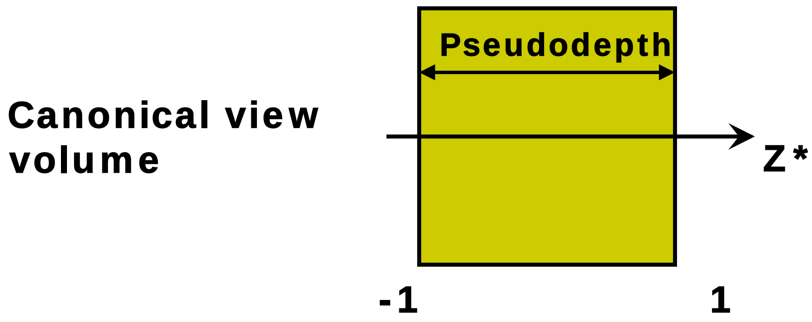
Perspective Transformation using Pseudodepth

$$(x^*, y^*, z^*) = \left(x\frac{N}{-z}, y\frac{N}{-z}, \frac{az + b}{-z} \right)$$

- Choose ***a, b*** so as z varies from **Near** to **Far**, pseudodepth varies from **–1** to **1** (canonical cube)

- Boundary conditions
  - z* = -1 when z = -N
  - z* = 1 when z = -F





Transformation of z: Solve for a and b

- Solving:

$$z^* = \frac{az + b}{-z}$$

- Use boundary conditions
  - z* = -1 when z = -N.........(1)
  - z* = 1 when z = -F...........(2)
- Set up simultaneous equations

$$-1 = \frac{-aN + b}{N} \Rightarrow -N = -aN + b........(1)$$

$$1 = \frac{-aF + b}{F} \Rightarrow F = -aF + b........(2)$$

Transformation of z: Solve for a and b

$$-N = -aN + b ....(1)$$

$$F = -aF + b ....(2)$$

- Multiply both sides of (1) by -1

$$N = aN - b ....(3)$$

- Add eqns (2) and (3)

$$F + N = aN - aF$$

$$\Rightarrow a = \frac{F+N}{N-F} = \frac{-(F+N)}{F-N} ....(4)$$

- Now put (4) back into (3)

Transformation of z: Solve for a and b

- Put solution for *a* back into eqn (3)

$$N = aN - b........(3)$$

$$\Rightarrow N = \frac{-N(F+N)}{F-N} - b$$

$$\Rightarrow b = -N - \frac{-N(F+N)}{F-N}$$

$$\Rightarrow b = \frac{-N(F-N) - N(F+N)}{F-N} = \frac{-NF - N^2 - NF + N^2}{F-N} = \frac{-2NF}{F-N}$$

- So

$$a = \frac{-(F+N)}{F-N} \qquad b = \frac{-2FN}{F-N}$$

What does this mean?

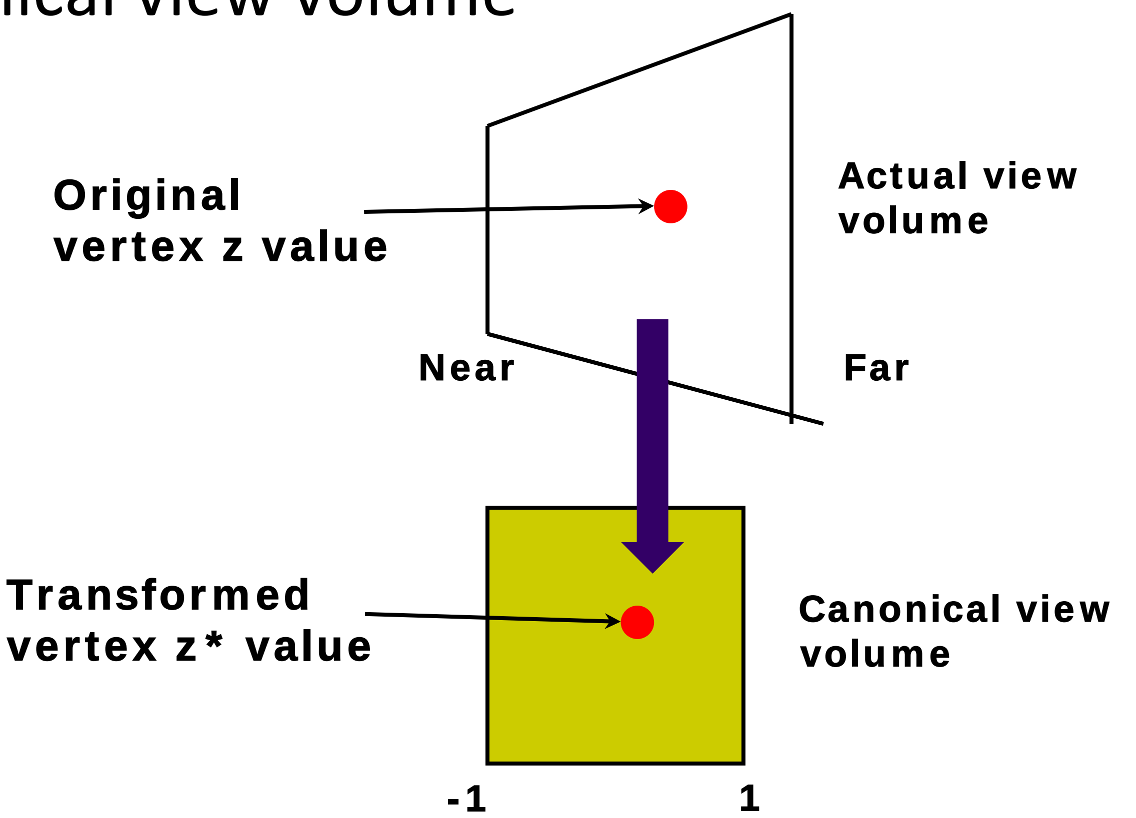
- Original point z in original view volume, transformed into z* in canonical view volume

$$z^* = \frac{az + b}{-z}$$

- where

$$a = \frac{-(F+N)}{F-N}$$

$$b = \frac{-2FN}{F-N}$$



Homogenous Coordinates

- Want to express projection transform as 4x4 matrix
- Previously, homogeneous coordinates of

  P = (Px,Py,Pz) => (Px,Py,Pz,1)
- Introduce arbitrary scaling factor, w, so that

  P = (wPx, wPy, wPz, w) (**Note:** w is non-zero)
- For example, the point P = (2,4,6) can be expressed as
  - (2,4,6,1)
  - or (4,8,12,2) where w=2
  - or (6,12,18,3) where w = 3, or….
- To convert from homogeneous back to ordinary coordinates, first divide all four terms by **w** and discard 4^{th} term

Perspective Projection Matrix

- Recall Perspective Transform

$$(x^*, y^*, z^*) = \left(x\frac{N}{-z}, y\frac{N}{-z}, \frac{az + b}{-z} \right)$$

- We have: $x^* = x\frac{N}{-z}$ $\quad y^* = y\frac{N}{-z}$ $\quad z^* = \frac{az+b}{-z}$

- In matrix form:

$$\begin{pmatrix} N & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} wx \\ wy \\ wz \\ w \end{pmatrix} = \begin{pmatrix} wNx \\ wNy \\ w(az + b) \\ -wz \end{pmatrix} \Rightarrow \begin{pmatrix} x\frac{N}{-z} \\ y\frac{N}{-z} \\ \frac{az+b}{-z} \\ 1 \end{pmatrix}$$

Perspective Transform Matrix | **Original vertex** | **Transformed Vertex** | **Transformed Vertex after dividing by 4th term**

Perspective Projection Matrix

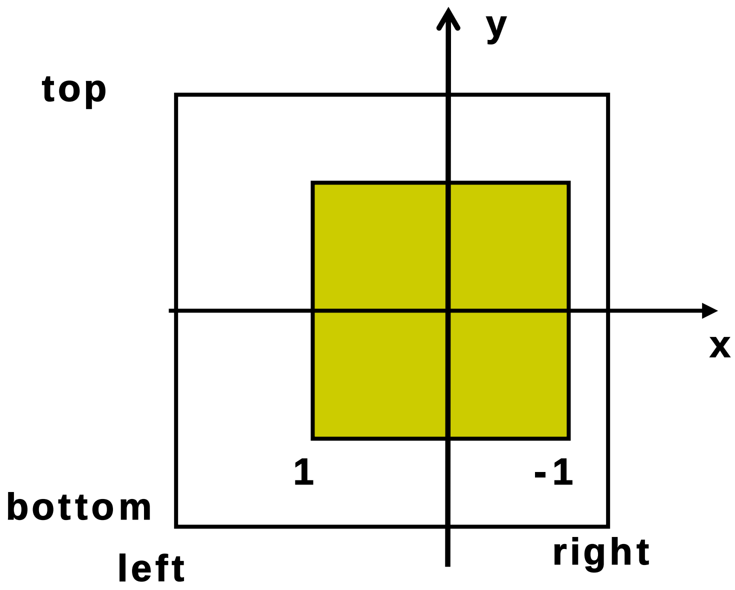
$$\begin{pmatrix} N & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} wP_x \\ wP_y \\ wP_z \\ w \end{pmatrix} = \begin{pmatrix} wNP_x \\ wNP_y \\ w(aP_z + b) \\ -wP_z \end{pmatrix} \Rightarrow \begin{pmatrix} x\dfrac{N}{-z} \\ y\dfrac{N}{-z} \\ \dfrac{az+b}{-z} \\ 1 \end{pmatrix}$$

$$a = \frac{-(F+N)}{F-N} \qquad b = \frac{-2FN}{F-N}$$

- In perspective transform matrix, already solved for ***a*** and ***b***:
- So, we have transform matrix to transform **z** values

Perspective Projection

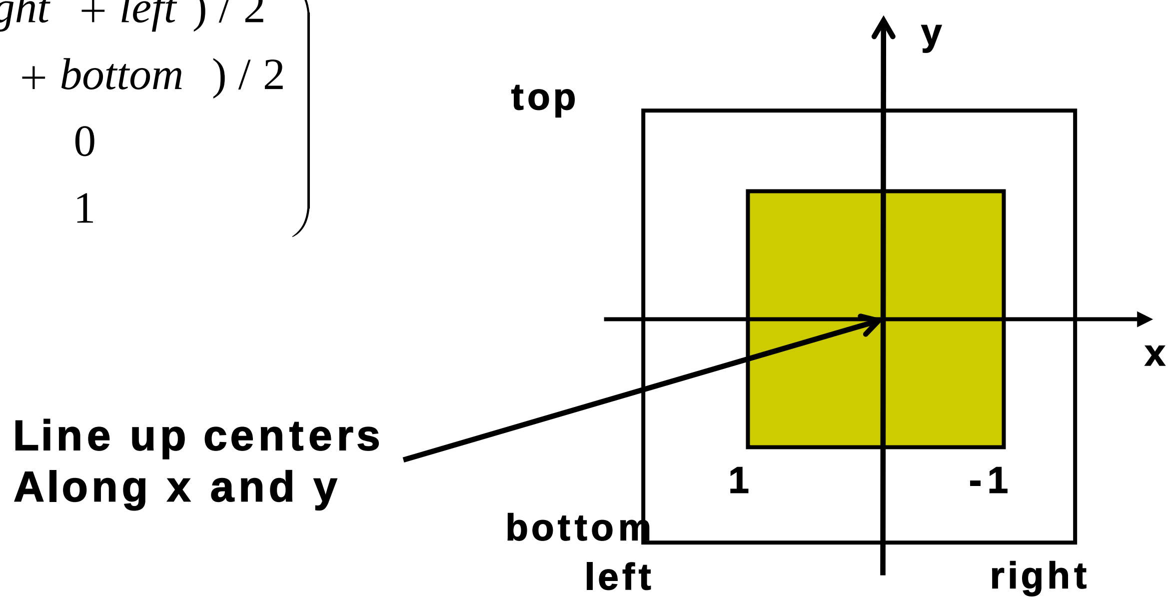
- Not done yet!! Can now transform z!
- Also need to transform the **x** = (left, right) and **y** = (bottom, top) ranges of viewing frustum to [-1, 1]
- Similar to glOrtho, we need to translate and scale previous matrix along x and y to get final projection transform matrix
- we translate by
  - –(right + left)/2 in x
  - -(top + bottom)/2 in y
- Scale by:
  - 2/(right – left) in x
  - 2/(top – bottom) in y



Perspective Projection

- Translate along x and y to line up center with origin of CVV
  - –(right + left)/2 in x
  - -(top + bottom)/2 in y
- Multiply by translation matrix:

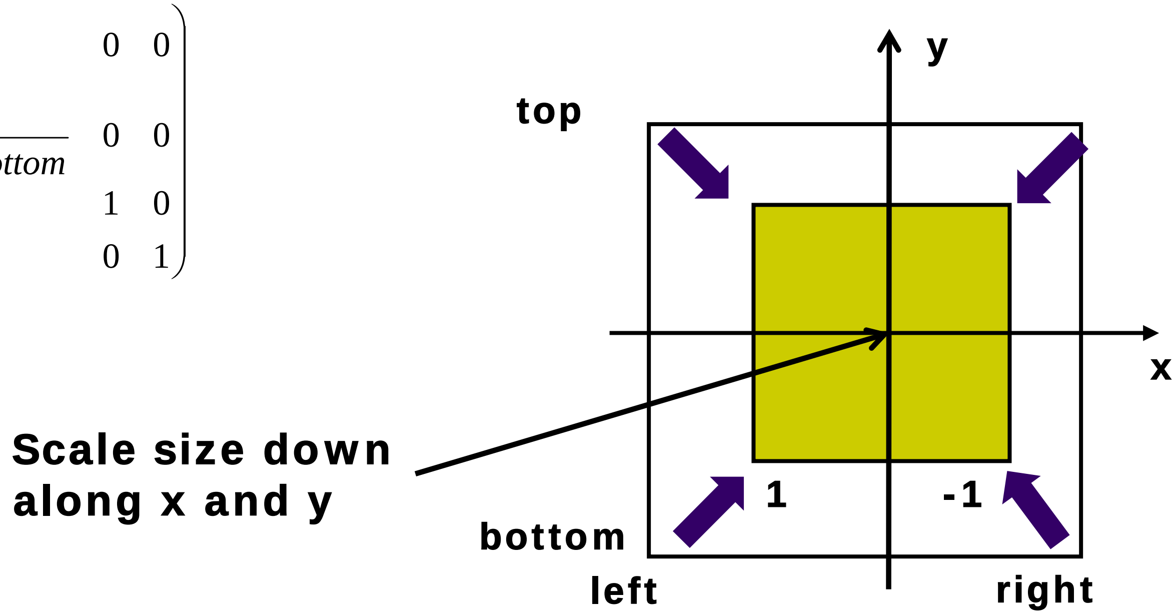
$$\begin{pmatrix} 1 & 0 & 0 & -(right + left)/2 \\ 0 & 1 & 0 & -(top + bottom)/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Perspective Projection

- To bring view volume size down to size of of CVV, scale by
  - 2/(right – left) in x
  - 2/(top – bottom) in y
- Multiply by scale matrix:

$$\begin{pmatrix} \frac{2}{right-left} & 0 & 0 & 0 \\ 0 & \frac{2}{top-bottom} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Perspective Projection Matrix

Scale · **Translate** · **Previous Perspective Transform Matrix**

$$\begin{pmatrix} \frac{2}{right-left} & 0 & 0 & 0 \\ 0 & \frac{2}{top-bottom} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & -(right+left)/2 \\ 0 & 1 & 0 & -(top+bottom)/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

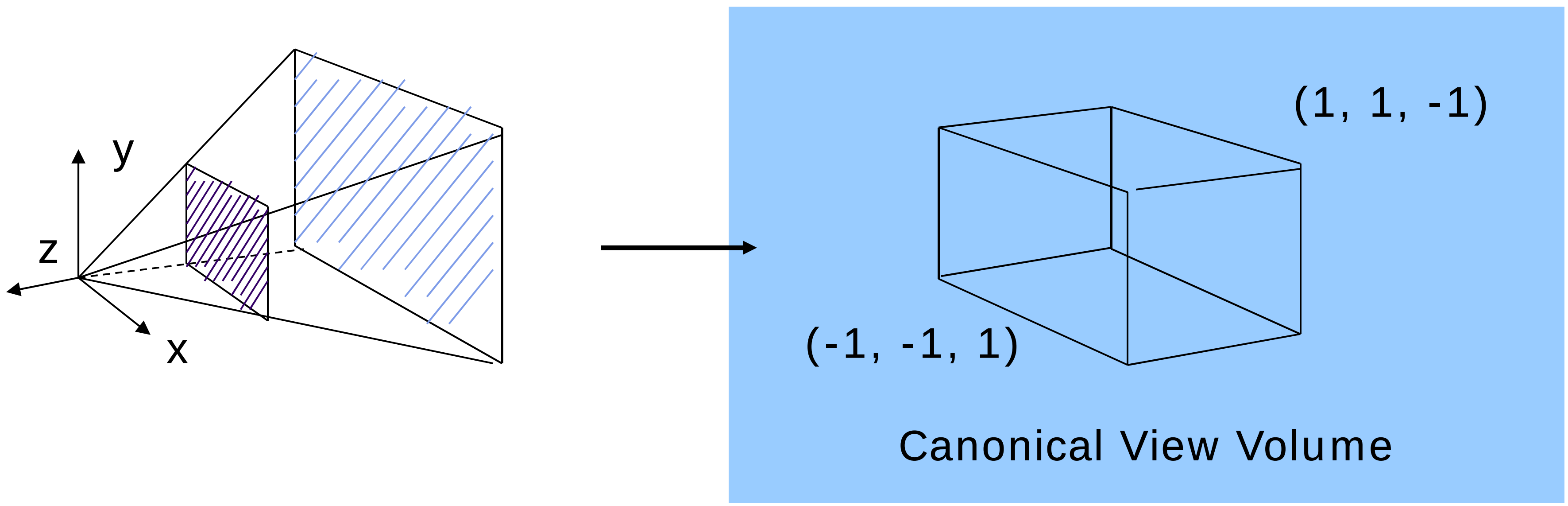
$$\Rightarrow \begin{pmatrix} \frac{2N}{x\max - x\min} & 0 & \frac{right+left}{right-left} & 0 \\ 0 & \frac{2N}{top-bottom} & \frac{top+bottom}{top-bottom} & 0 \\ 0 & 0 & \frac{-(F+N)}{F-N} & \frac{-2FN}{F-N} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Final Perspective Transform Matrix

glFrustum(left, right, bottom, top, N, F) N = near plane, F = far plane

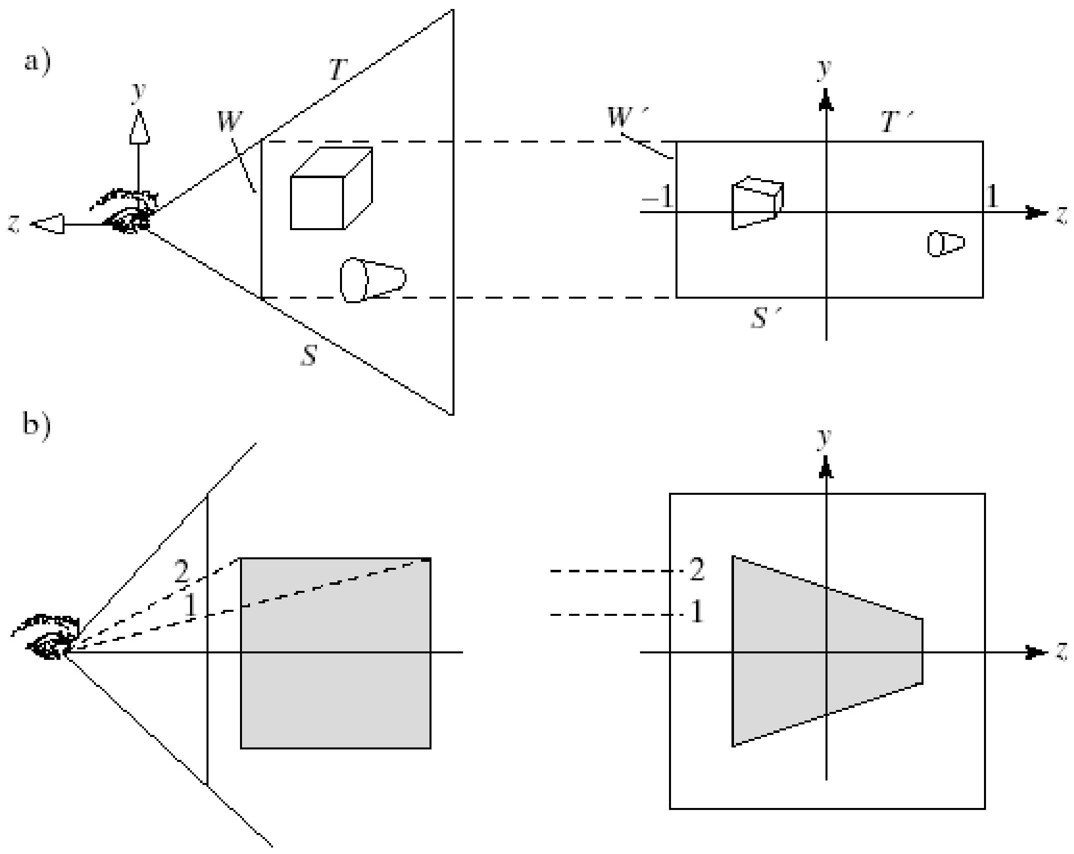
Perspective Transformation

- After perspective transformation, viewing frustum volume is transformed into canonical view volume

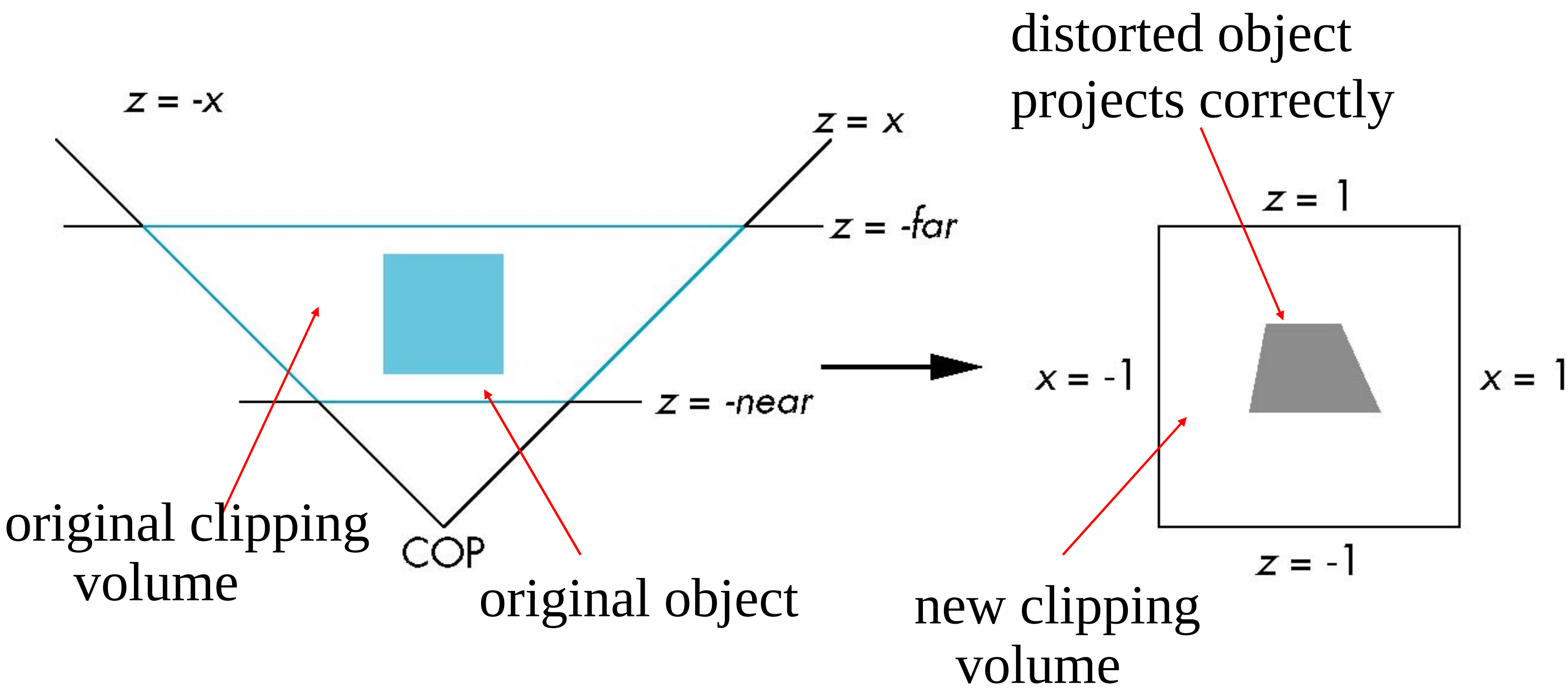


Geometric Nature of Perspective Transform

a) Lines through eye map into lines parallel to z axis after transform
b) Lines perpendicular to z axis map to lines perp to z axis after transform



Normalization Transformation



Implementation

- Set modelview and projection matrices in application program
- Pass matrices to shader

```
void display( ){
    .....
    model_view = LookAt(eye, at, up);
    projection = Ortho(left, right, bottom,top, near, far);

    // pass model_view and projection matrices to shader
    glUniformMatrix4fv(matrix_loc, 1, GL_TRUE, model_view);
    glUniformMatrix4fv(projection_loc, 1, GL_TRUE, projection);
 .....
}
```

Implementation

- And the corresponding shader

```
in vec4 vPosition;
in vec4 vColor;
Out vec4 color;
uniform mat4 model_view;
Uniform mat4 projection;

void main( )
{
   gl_Position = projection*model_view*vPosition;
   color = vColor;
}
```

References

- Interactive Computer Graphics (6th edition), Angel and Shreiner
- Computer Graphics using OpenGL (3rd edition), Hill and Kelley