Computer Graphics 543
Lecture 5a: Rotations and Matrix Concatenation

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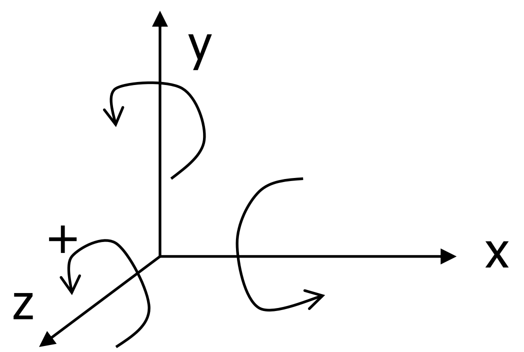
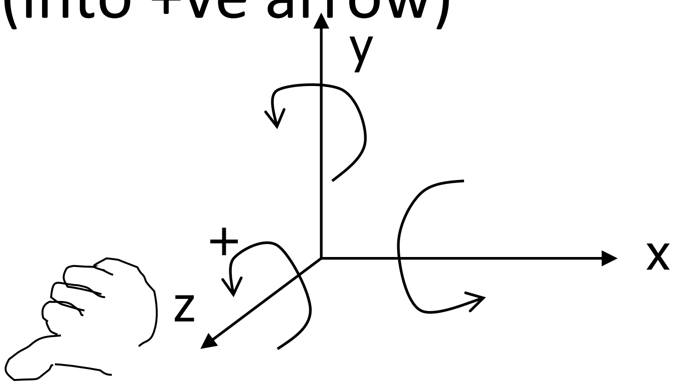
Rotating in 3D

- Many degrees of freedom. Rotate about what axis?
- 3D rotation: about a defined axis
- Different transform matrix for:
  - Rotation about x-axis
  - Rotation about y-axis
  - Rotation about z-axis

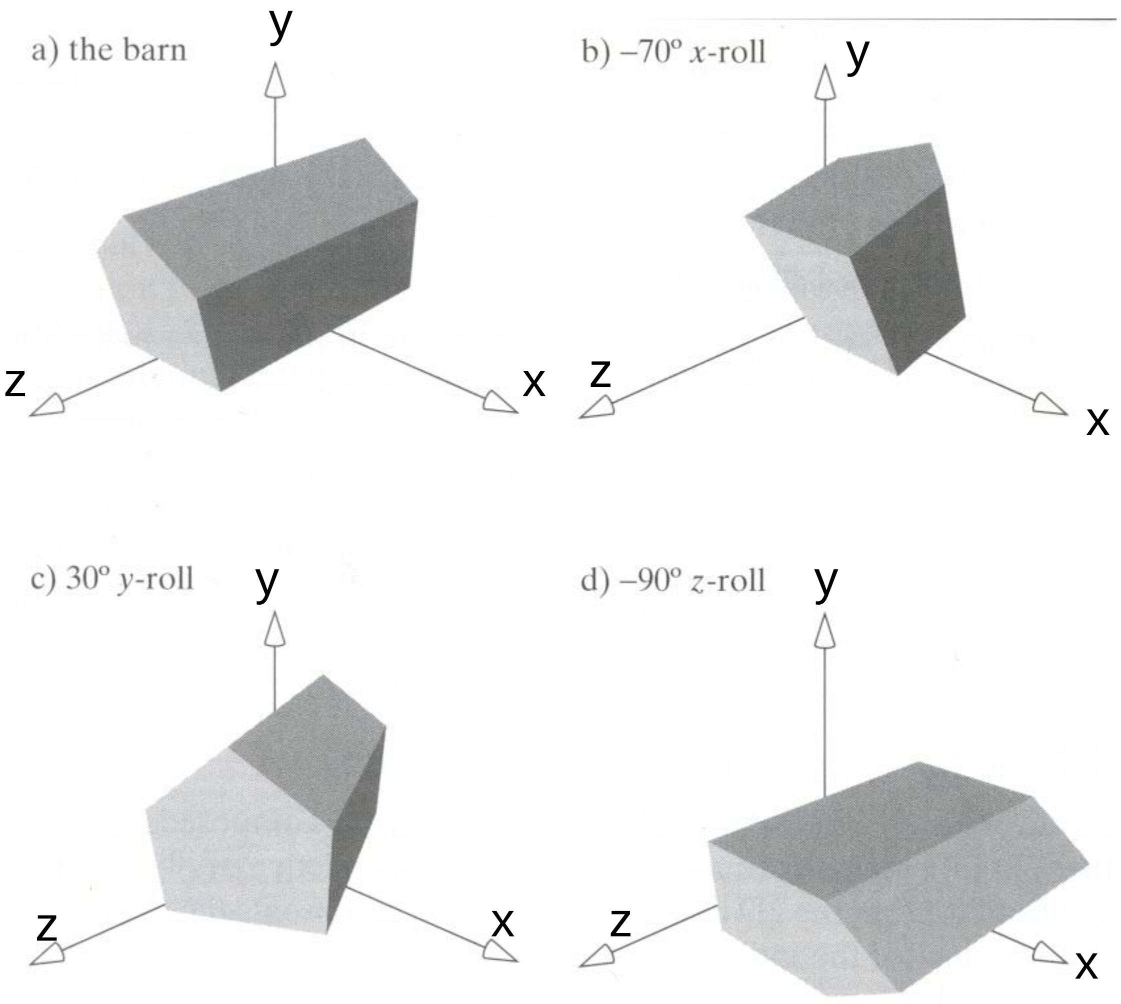


Rotating in 3D

- New terminology
  - **X-roll:** rotation about x-axis
  - **Y-roll:** rotation about y-axis
  - **Z-roll:** rotation about z-axis
- Which way is +ve rotation
  - Look in –ve direction (into +ve arrow)
  - CCW is +ve rotation



Rotating in 3D



Rotating in 3D

- For a rotation angle, β about an axis
- Define:

$$c = \cos(\beta) \qquad s = \sin(\beta)$$

x-roll or (RotateX)

$$R_x(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotating in 3D

y-roll (or RotateY)

$$R_y(\beta) = \begin{pmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rules:

•Write 1 in rotation row, column

•Write 0 in the other rows/columns

•Write c,s in rect pattern

z-roll (or RotateZ)

$$R_z(\beta) = \begin{pmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: Rotating in 3D

Question: Using **y-roll** equation, rotate *P* = (3,1,4) by 30 degrees:

Answer: c = cos(30) = 0.866, s = sin(30) = 0.5, and

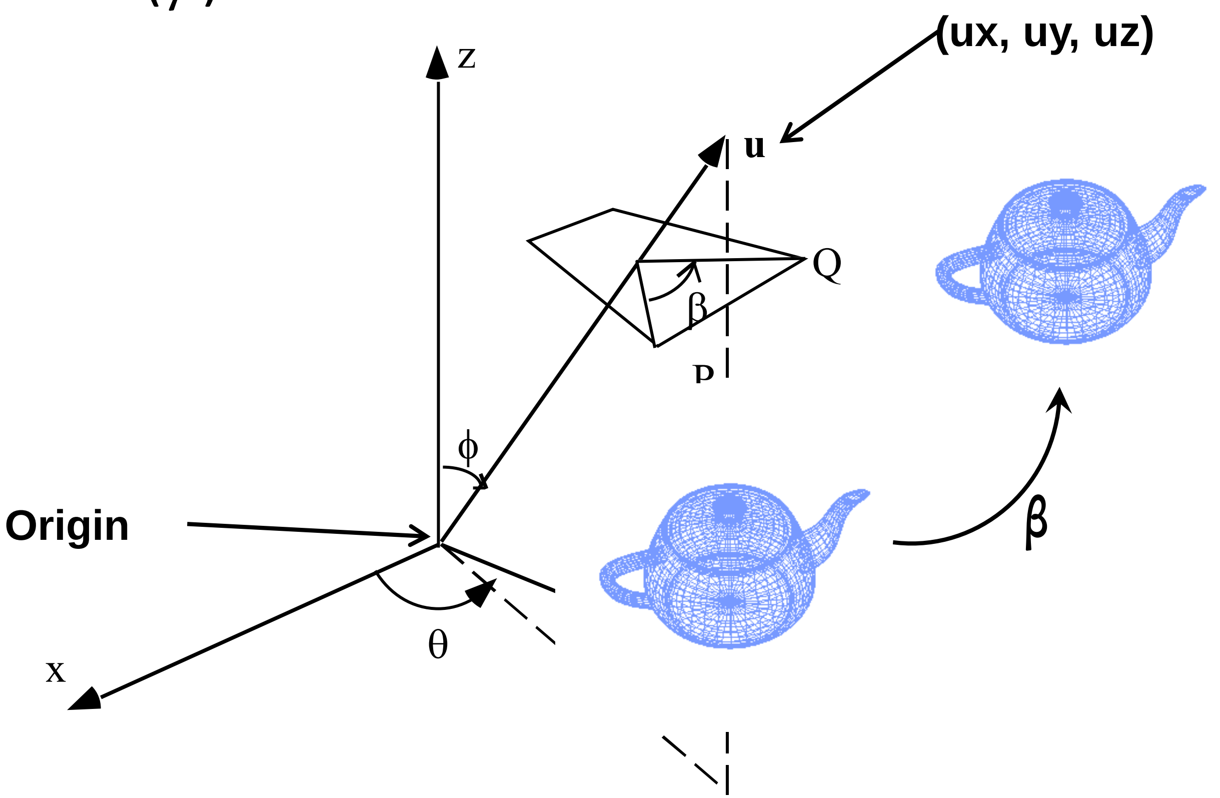
$$Q = \begin{pmatrix} c & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ -s & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4.6 \\ 1 \\ 1.964 \\ 1 \end{pmatrix}$$

Line 1: (3 x c) + (1 x 0) + (4 x s) + (1 x 0)

= (3 x 0.866) + (4 x 0.5) = 4.6

3D Rotation

- **Rotate(angle, ux, uy, uz):** rotate by angle β about an **arbitrary** axis (a vector) passing through **origin** and **(ux, uy, uz)**
- **Note:** Angular position of **u** specified as azimuth/longitude (θ) and latitude (ϕ)



Approach 1: 3D Rotation About Arbitrary Axis

- Can compose arbitrary rotation as combination of:
  - X-roll (by an angle β_1)
  - Y-roll (by an angle β_2)
  - Z-roll (by an angle β_3)

$$M = R_z(\beta_3)R_y(\beta_2)R_x(\beta_1)$$

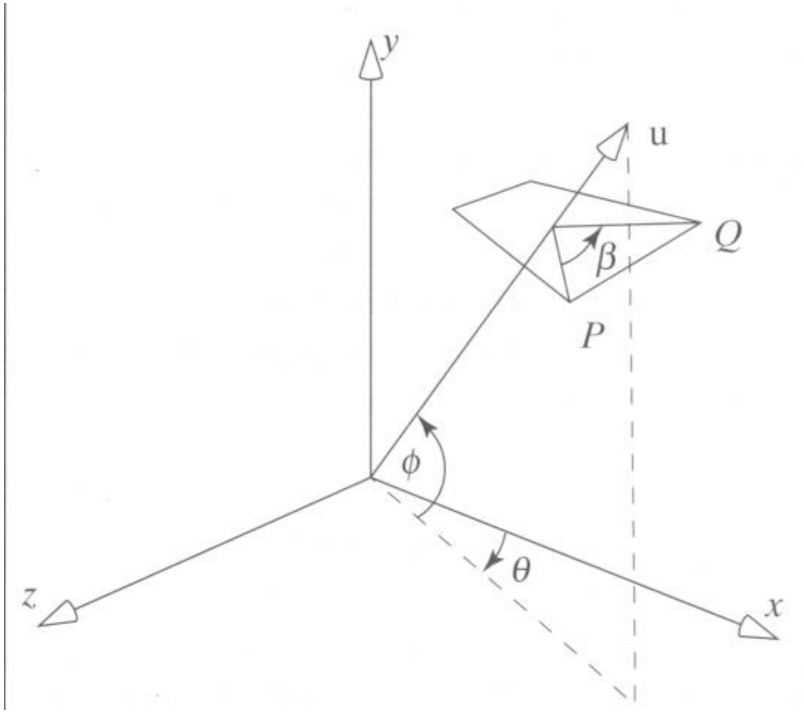
Read in reverse order

Approach 1: 3D Rotation using Euler Theorem

- **Classic:** use Euler's theorem
- **Euler's theorem:** any sequence of rotations = one rotation about some axis
- Want to rotate β about arbitrary axis **u** through origin
- Our approach:
  1. Use two rotations to align **u** and **x-axis**
  2. Do **x-roll** through angle β
  3. Negate two previous rotations to de-align **u** and **x-axis**

Approach 1: 3D Rotation using Euler Theorem

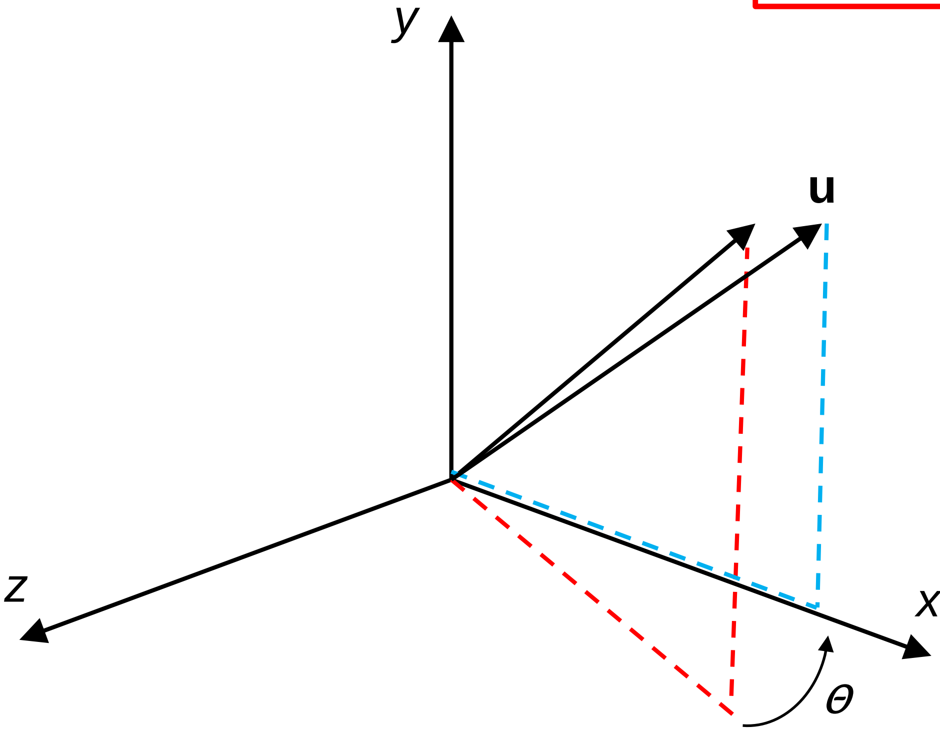
- **Note:** Angular position of **u** specified as azimuth (θ) and latitude (ϕ)
- First try to align **u** with x axis



Approach 1: 3D Rotation using Euler Theorem

- **Step 1:** Do y-roll to line up rotation axis with x-y plane

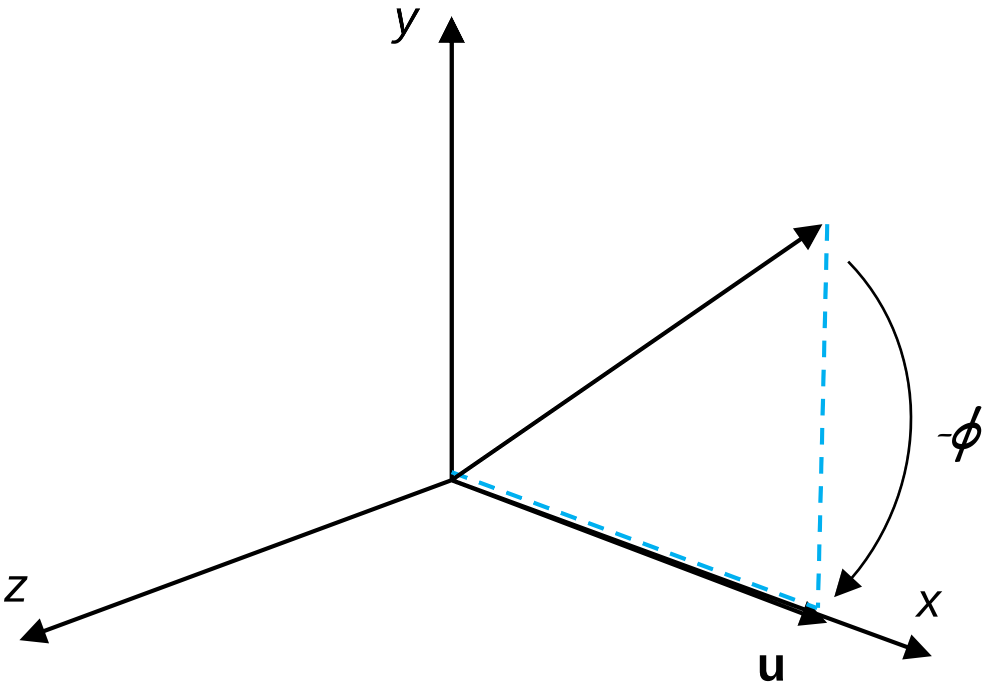
$$\boxed{R_y(\theta)}$$



Approach 1: 3D Rotation using Euler Theorem

- **Step 2:** Do z-roll to line up rotation axis with x axis

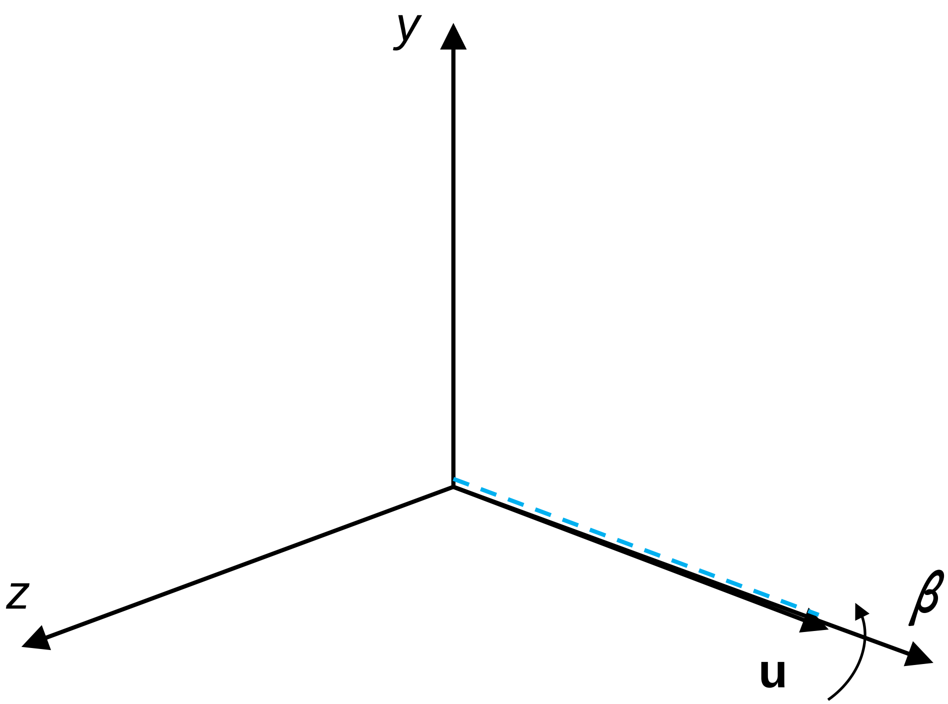
$$\boxed{R_z(-\phi)}R_y(\theta)$$

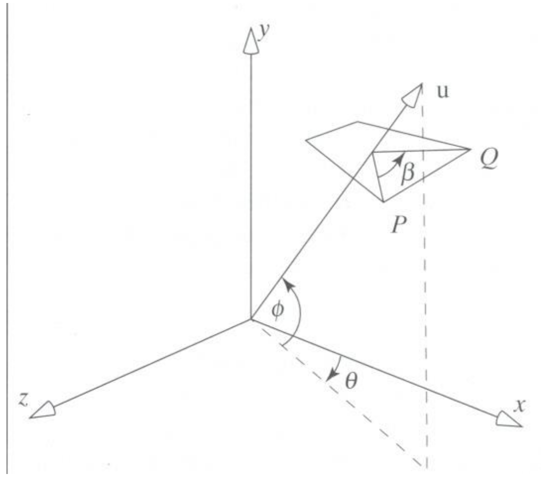


Approach 1: 3D Rotation using Euler Theorem

- **Remember:** Our goal is to do rotation by β around **u**
- But axis **u** is now lined up with x axis. So,
- **Step 3:** Do x-roll by β around axis **u**

$$\boxed{R_x(\beta)}R_z(-\phi)R_y(\theta)$$

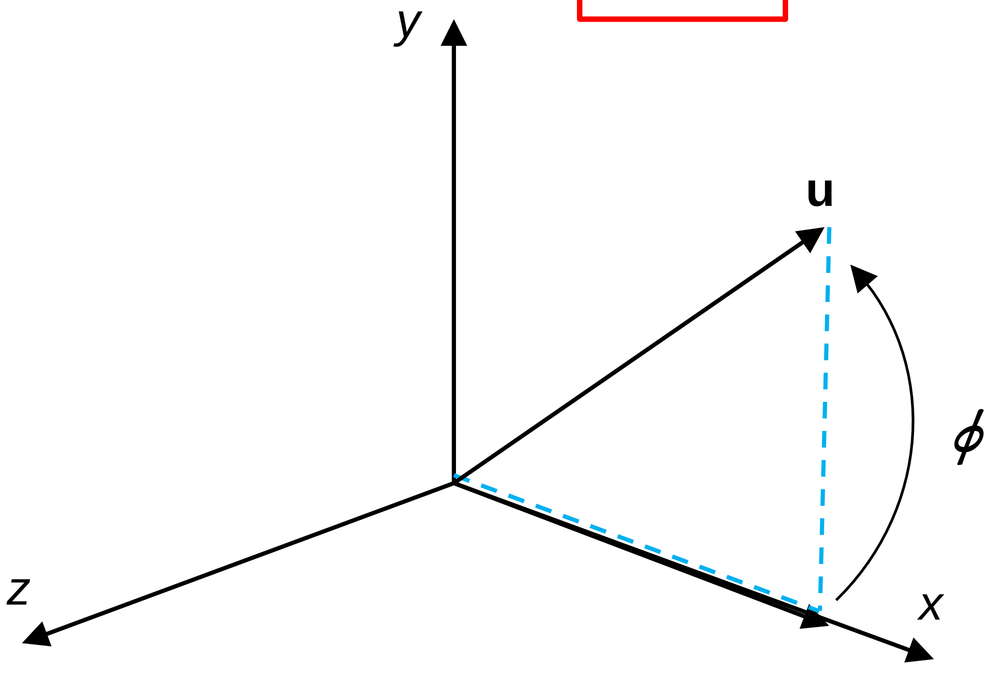




Approach 1: 3D Rotation using Euler Theorem

- Next 2 steps are to return vector **u** to original position
- **Step 4:** Do z-roll in x-y plane

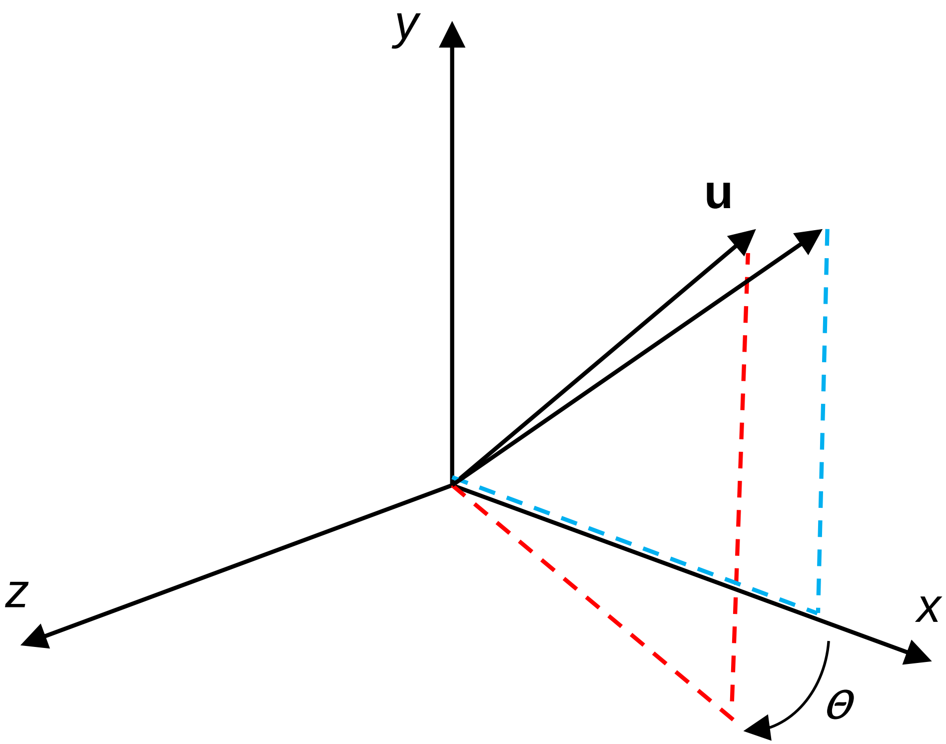
$$\boxed{R_z(\phi)}R_x(\beta)R_z(-\phi)R_y(\theta)$$

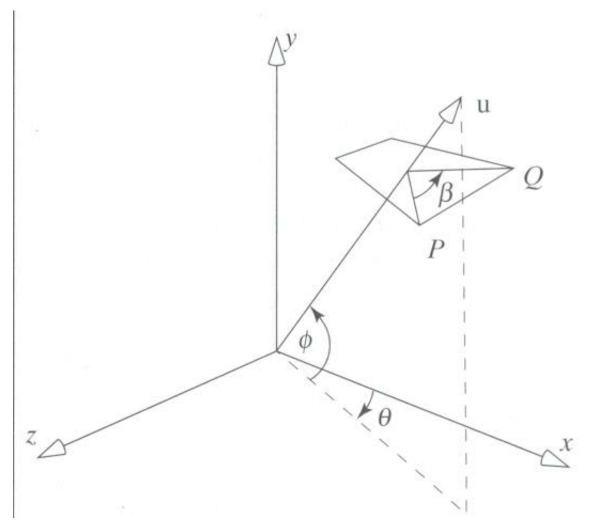


Approach 1: 3D Rotation using Euler Theorem

- **Step 5:** Do y-roll to return **u** to original position

$$R_u(\beta) = \boxed{R_y(-\theta)} R_z(\phi) R_x(\beta) R_z(-\phi) R_y(\theta)$$





Approach 2: Rotation using Quaternions

- Extension of imaginary numbers from 2 to 3 dimensions
- Requires 1 real and 3 imaginary components $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

- Quaternions can express rotations on sphere smoothly and efficiently

Approach 2: Rotation using Quaternions

- Derivation skipped! Check answer
- Solution has lots of symmetry

$$R(\beta) = \begin{pmatrix} c + (1-c)\mathbf{u}_x^2 & (1-c)\mathbf{u}_y\mathbf{u}_x + s\mathbf{u}_z & (1-c)\mathbf{u}_z\mathbf{u}_x + s\mathbf{u}_y & 0 \\ (1-c)\mathbf{u}_x\mathbf{u}_y + s\mathbf{u}_z & c + (1-c)\mathbf{u}_y^2 & (1-c)\mathbf{u}_z\mathbf{u}_y - s\mathbf{u}_x & 0 \\ (1-c)\mathbf{u}_x\mathbf{u}_z - s\mathbf{u}_y & (1-c)\mathbf{u}_y\mathbf{u}_z - s\mathbf{u}_x & c + (1-c)\mathbf{u}_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$c = \cos(\beta)$ $s = \sin(\beta)$ Arbitrary axis **u**

Inverse Matrices

- Can compute inverse matrices by general formulas
- But some easy **inverse transform** observations
  - Translation: $\mathbf{T}^{-1}(d_x, d_y, d_z) = \mathbf{T}(-d_x, -d_y, -d_z)$
  - Scaling: $\mathbf{S}^{-1}(s_x, s_y, s_z) = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$
  - Rotation: $\mathbf{R}^{-1}(q) = \mathbf{R}(-q)$
    - Holds for any rotation matrix

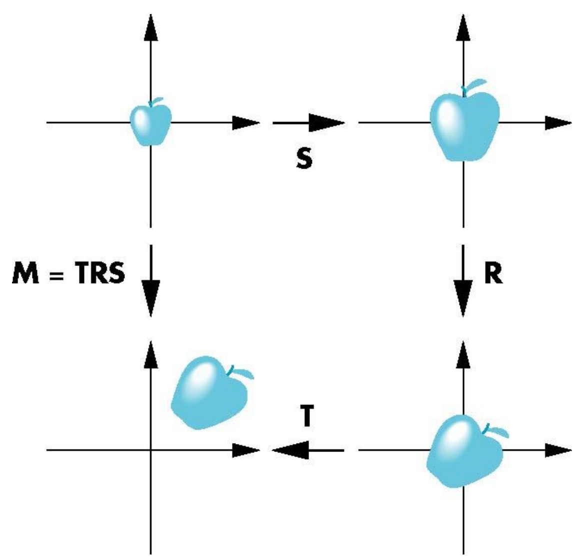
Instancing

- During modeling, often start with simple object centered at origin, aligned with axis, and unit size
- Can declare one copy of each shape in scene
- E.g. declare 1 mesh for soldier, 500 instances to create army
- Then apply *instance transformation* to its vertices to

  Scale

  Orient

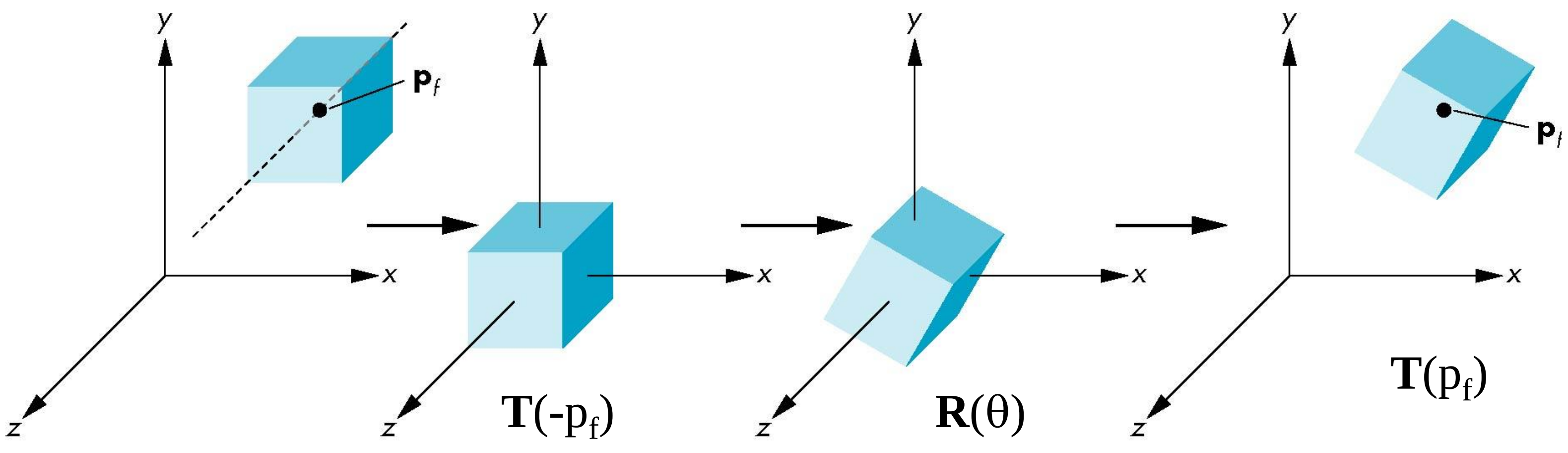
  Locate



Rotation About Arbitrary Point other than the Origin

- Default rotation matrix is about origin
- How to rotate about any arbitrary point p_f (Not origin)?
  - Move fixed point to origin $\mathbf{T}(-p_f)$
  - Rotate $\mathbf{R}(\theta)$
  - Move fixed point back $\mathbf{T}(p_f)$

So, $\mathbf{M} = \mathbf{T}(p_f)\, \mathbf{R}(\theta)\, \mathbf{T}(-p_f)$



Scale about Arbitrary Center

- Similary, default scaling is about origin
- To scale about arbitrary point P = (Px, Py, Pz) by (Sx, Sy, Sz)
  1. **Translate** object by T(-Px, -Py, -Pz) so P coincides with origin
  2. **Scale** object by (Sx, Sy, Sz)
  3. **Translate** object back: T(Px, Py, Py)
- In matrix form: T(Px,Py,Pz) (Sx, Sy, Sz) T(-Px,-Py,-Pz) * P

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & Px \\ 0 & 1 & 0 & Py \\ 0 & 0 & 1 & Pz \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -Px \\ 0 & 1 & 0 & -Py \\ 0 & 0 & 1 & -Pz \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Example

- Rotation about z axis by 30 degrees about a fixed point (1.0, 2.0, 3.0)

```
mat 4 m = Identity();
m = Translate(1.0, 2.0, 3.0)*
    Rotate(30.0, 0.0, 0.0, 1.0)*
    Translate(-1.0, -2.0, -3.0);
```

- Remember last matrix specified in program (i.e. translate matrix in example) is first applied

References

- Angel and Shreiner, Chapter 3
- Hill and Kelley, Computer Graphics Using OpenGL, 3rd edition