Computer Graphics (CS 543)
Lecture 2c: Fractals

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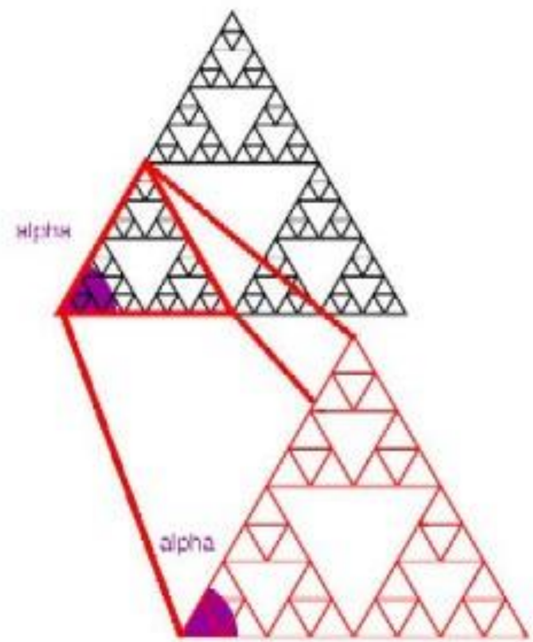
What are Fractals?

- Mathematical expressions to generate pretty pictures
- Evaluate math functions to create drawings
  - approach infinity -> converge to image
- Utilizes recursion on computers
- Popularized by Benoit Mandelbrot (Yale university)
- Dimensional:
  - Line is 1-dimensional
  - Plane is 2-dimensional
- Defined in terms of self-similarity

Fractals: Self-similarity

- See similar sub-images within image as we zoom in
- Example: surface roughness or profile same as we zoom in



Applications of Fractals

Grass

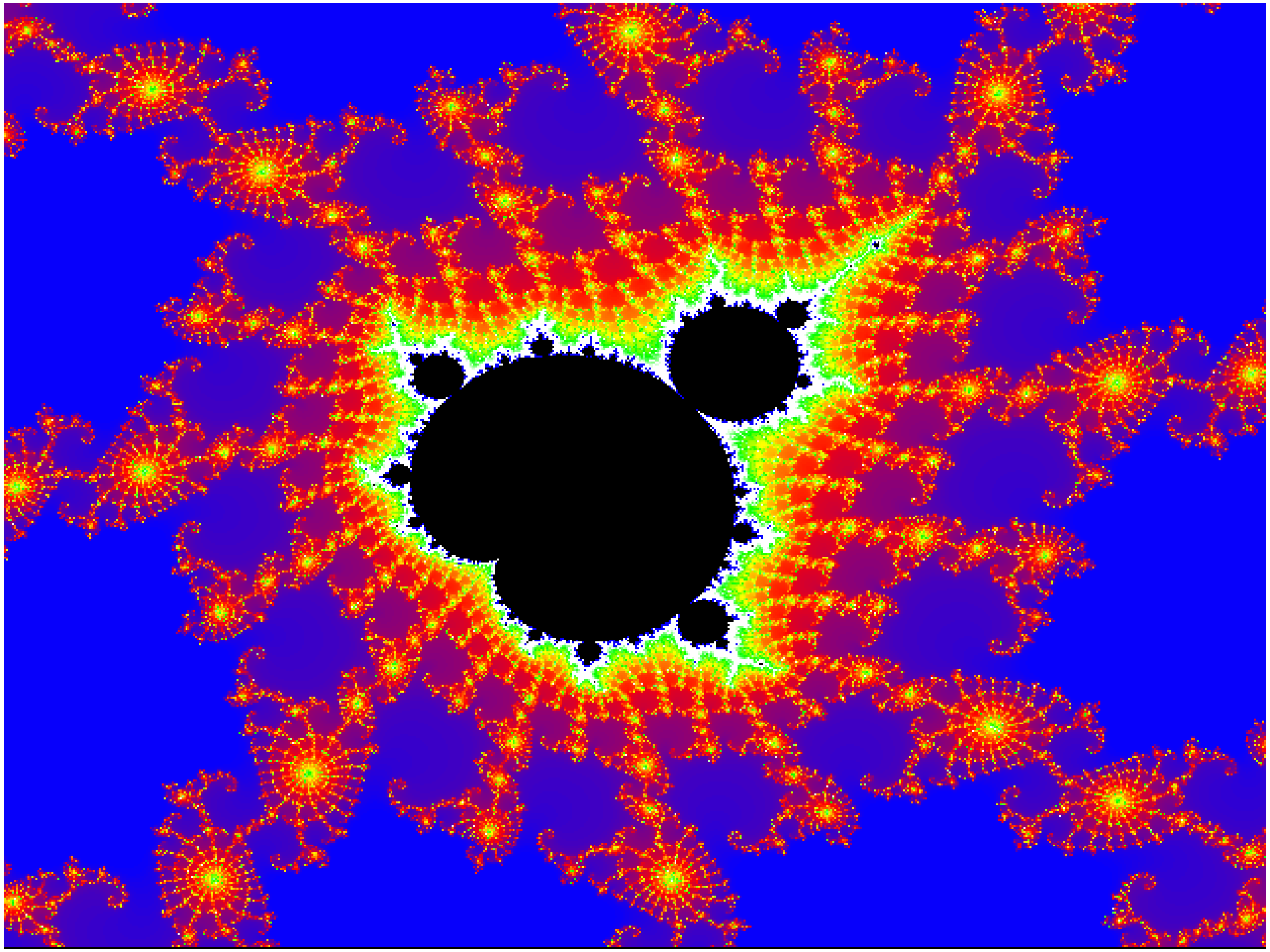
Coastline

Fire

- Other applications:
  - Mountains
  - Branches of a tree
  - Surface of a sponge
  - Cracks in the pavement
  - Designing antennae (www.fractenna.com)

Clouds

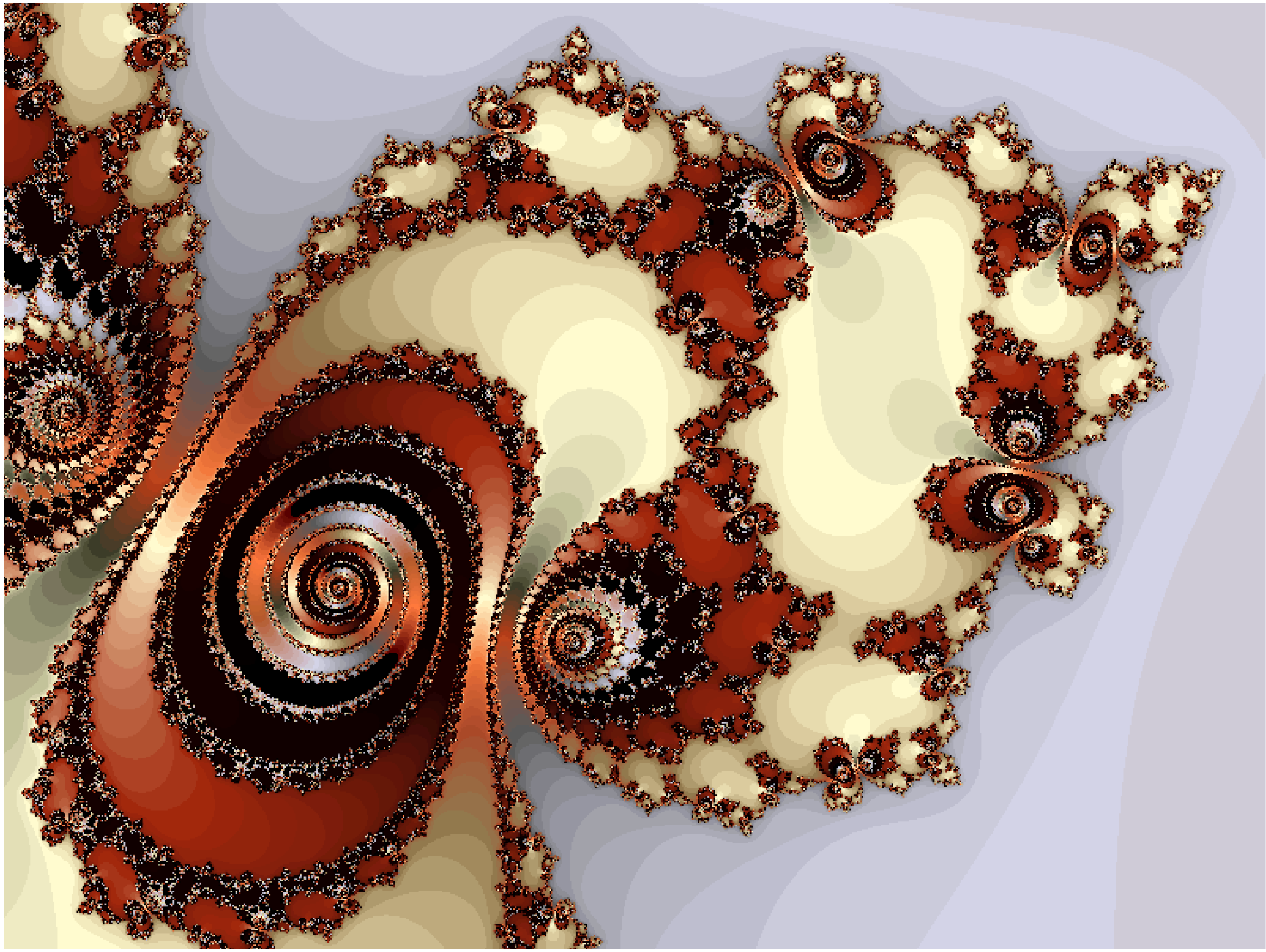
Example: Mandelbrot Set



Example: Fractal Terrain

Courtesy: Mountain 3D Fractal Terrain software

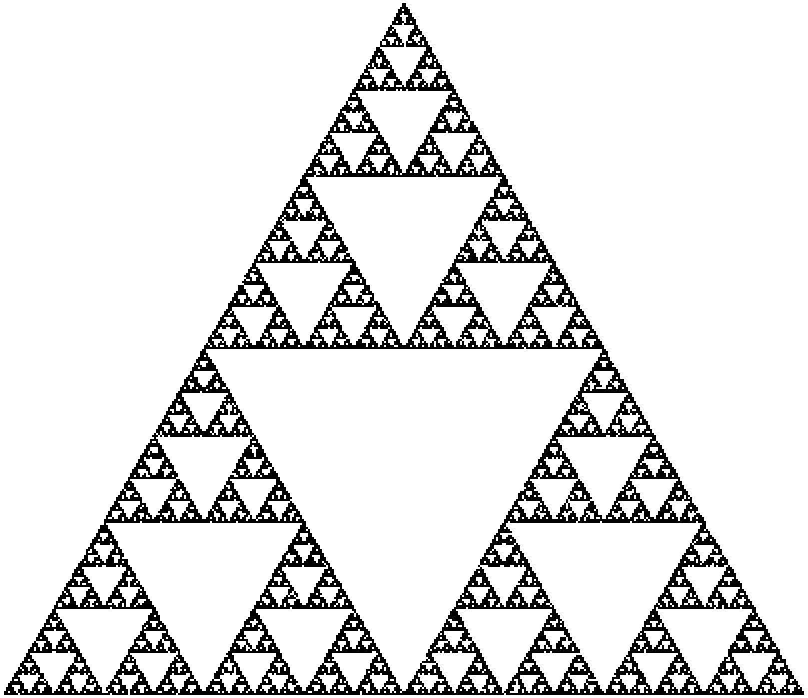
Application: Fractal Art



Courtesy: Internet Fractal Art Contest

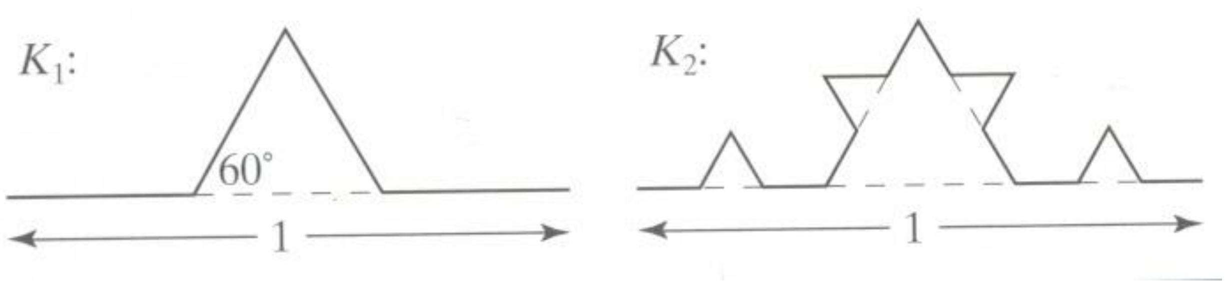
Recall: Sierpinski Gasket Program

- Popular fractal



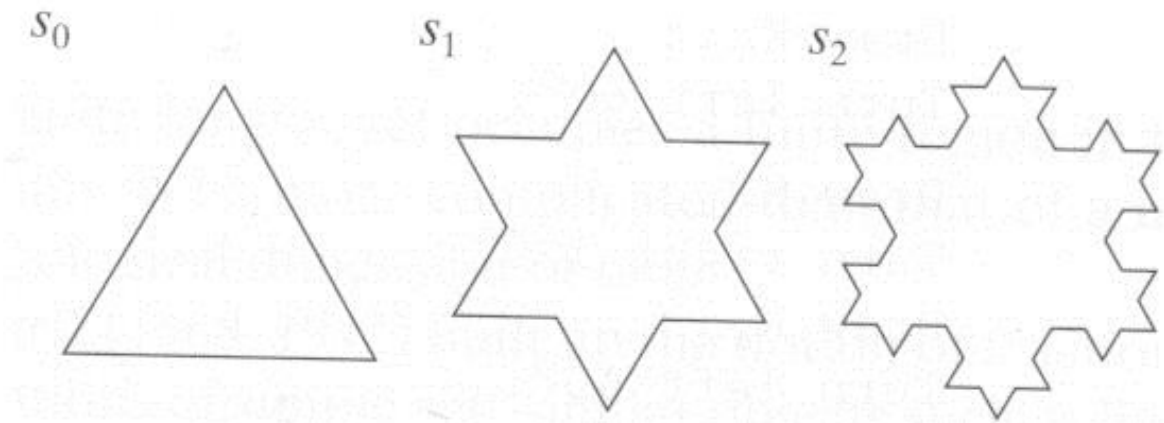
Koch Curves

- Discovered in 1904 by Helge von Koch
- Start with straight line of length 1
- Recursively:
  - Divide line into 3 equal parts
  - Replace middle section with triangular bump, sides of length 1/3
  - New length = 4/3

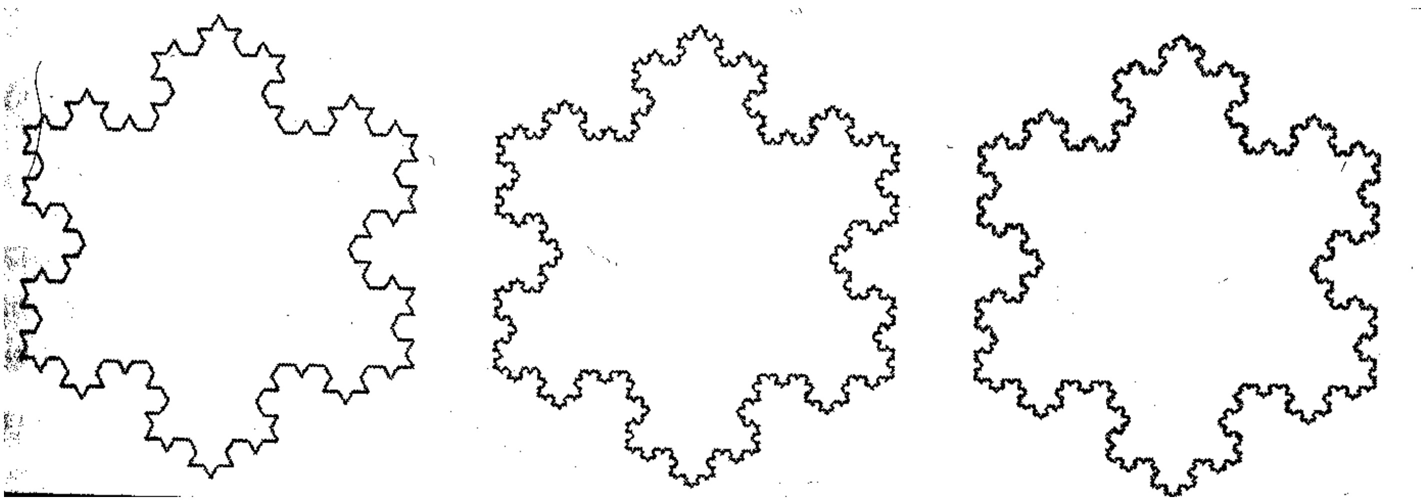


Koch Snowflakes

Can form Koch snowflake by joining three Koch curves



S_3, S_4, S_5,



Koch Snowflakes

Pseudocode, to draw K_n:

If (n equals 0) draw straight line

Else{

Draw K_{n-1}

Turn left 60°

Draw K_{n-1}

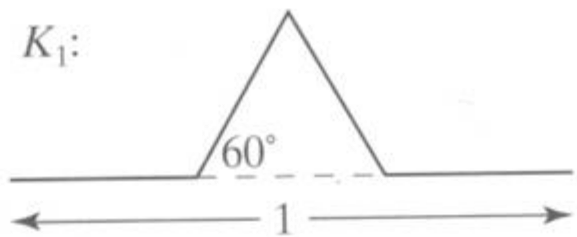
Turn right 120°

Draw K_{n-1}

Turn left 60°

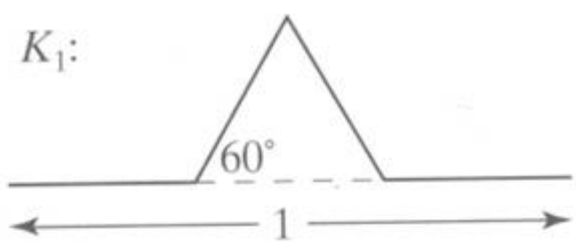
Draw K_{n-1}

}



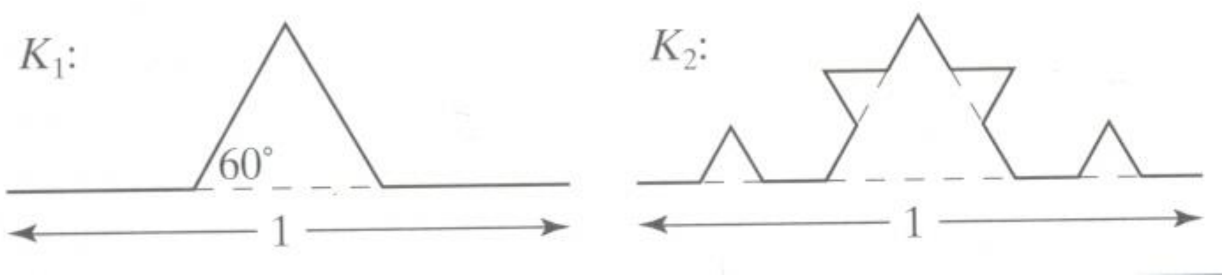
L-Systems: Lindenmayer Systems

- Express complex curves as simple set of **string-production** rules
- Example rules:
  - 'F': go forward a distance 1 in current direction
  - '+': turn right through angle ***A*** degrees
  - '-': turn left through angle ***A*** degrees
- Using these rules, can express koch curve as: "F-F++F-F"
- Angle ***A*** = 60 degrees



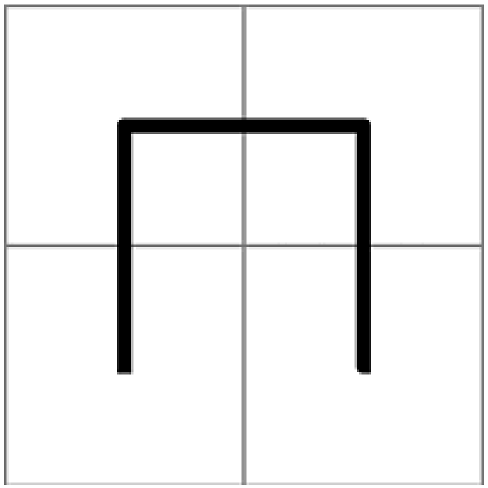
L-Systems: Koch Curves

- Rule for Koch curves is F -> F-F++F-F
- Means each iteration replaces every ‘F’ occurrence with “F-F++F-F”
- So, if initial string (called the **atom**) is ‘F’, then
- S_1 =“F-F++F-F”
- S_2 =“F-F++F-F- F-F++F-F++ F-F++F-F- F-F++F-F”
- S_3 = …..
- Gets very large quickly



Hilbert Curve

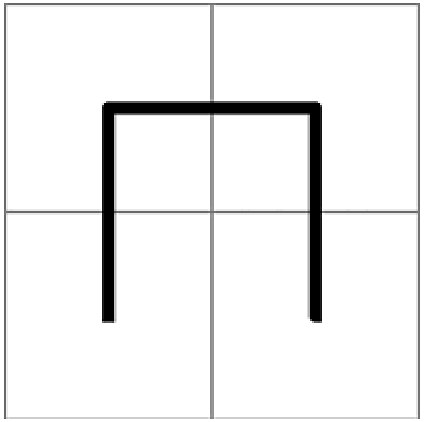
- Discovered by German Scientist, David Hilbert in late 1900s
- Space filling curve
- Drawn by connecting centers of 4 sub-squares, make up larger square.
- **Iteration 0:** 3 segments connect 4 centers in upside-down U

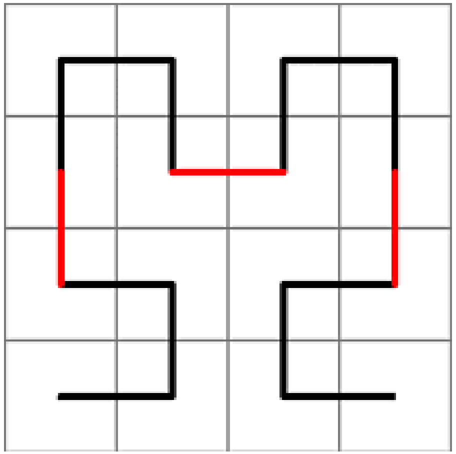


Iteration 0

Hilbert Curve: Iteration 1

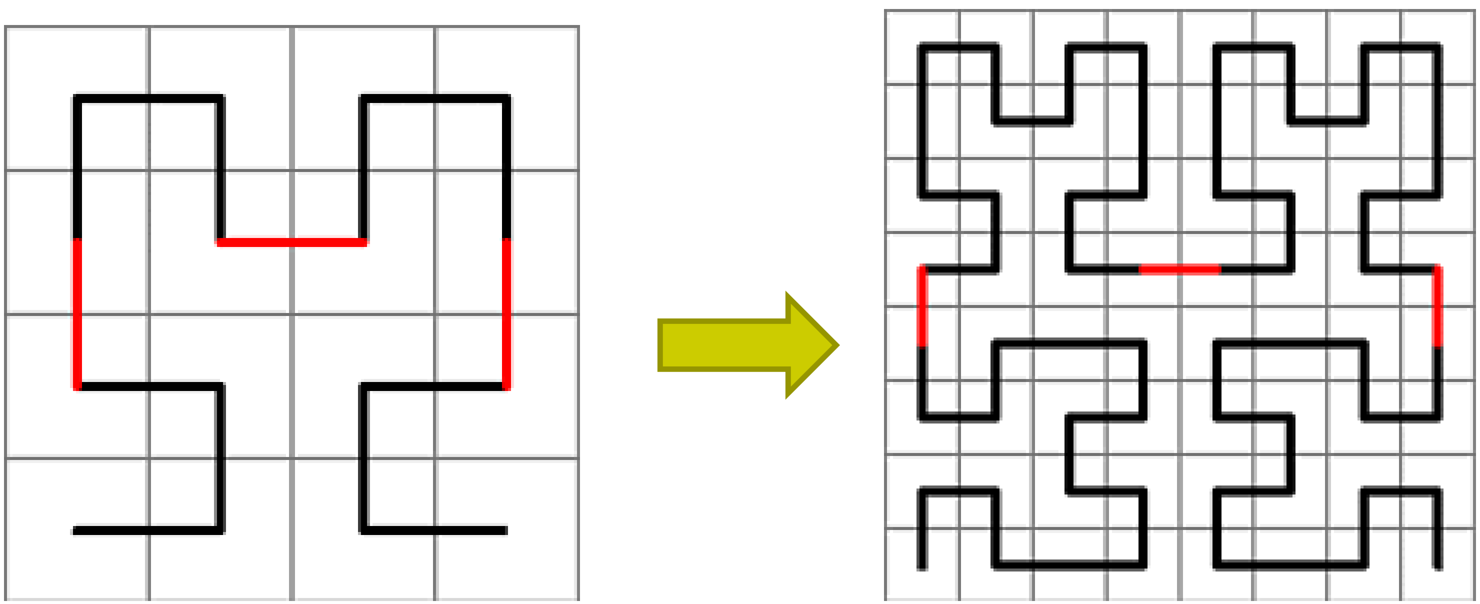
- Each of 4 squares divided into 4 more squares
- U shape shrunk to half its original size, copied into 4 sectors
- In top left, simply copied, top right: it's flipped vertically
- In the bottom left, rotated 90 degrees clockwise,
- Bottom right, rotated 90 degrees counter-clockwise.
- 4 pieces connected with 3 segments, each of which is same size as the shrunken pieces of the U shape (in red)





Hilbert Curve: Iteration 2

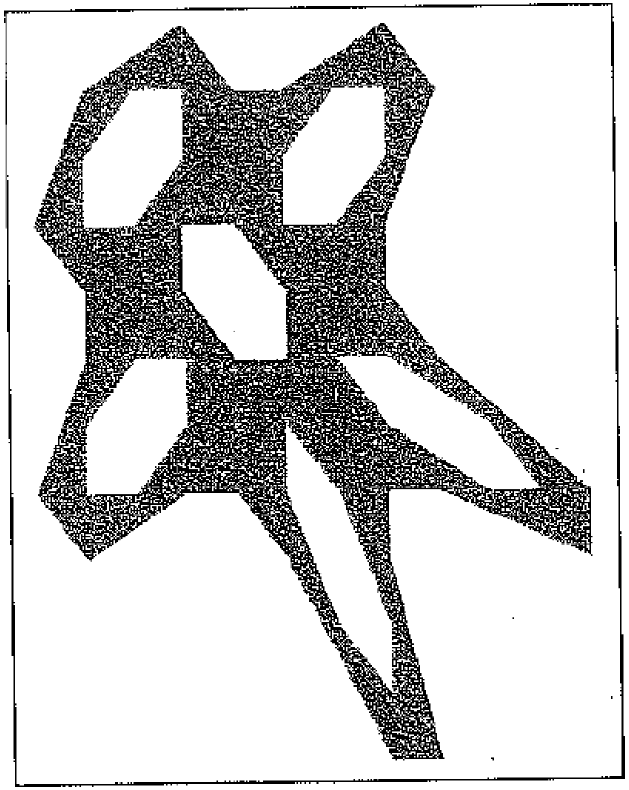
- Each of the 16 squares from iteration 1 divided into 4 squares
- Shape from iteration 1 shrunk and copied.
- 3 connecting segments (shown in red) are added to complete the curve.
- Implementation? Recursion is your friend!!



Gingerbread Man

- Each new point **q** is formed from previous point **p** using the equation

$$q.x = M(1 + 2L) - p.y + |p.x - LM|;$$
$$q.y = p.x.$$



- For 640 x 480 display area, use constants

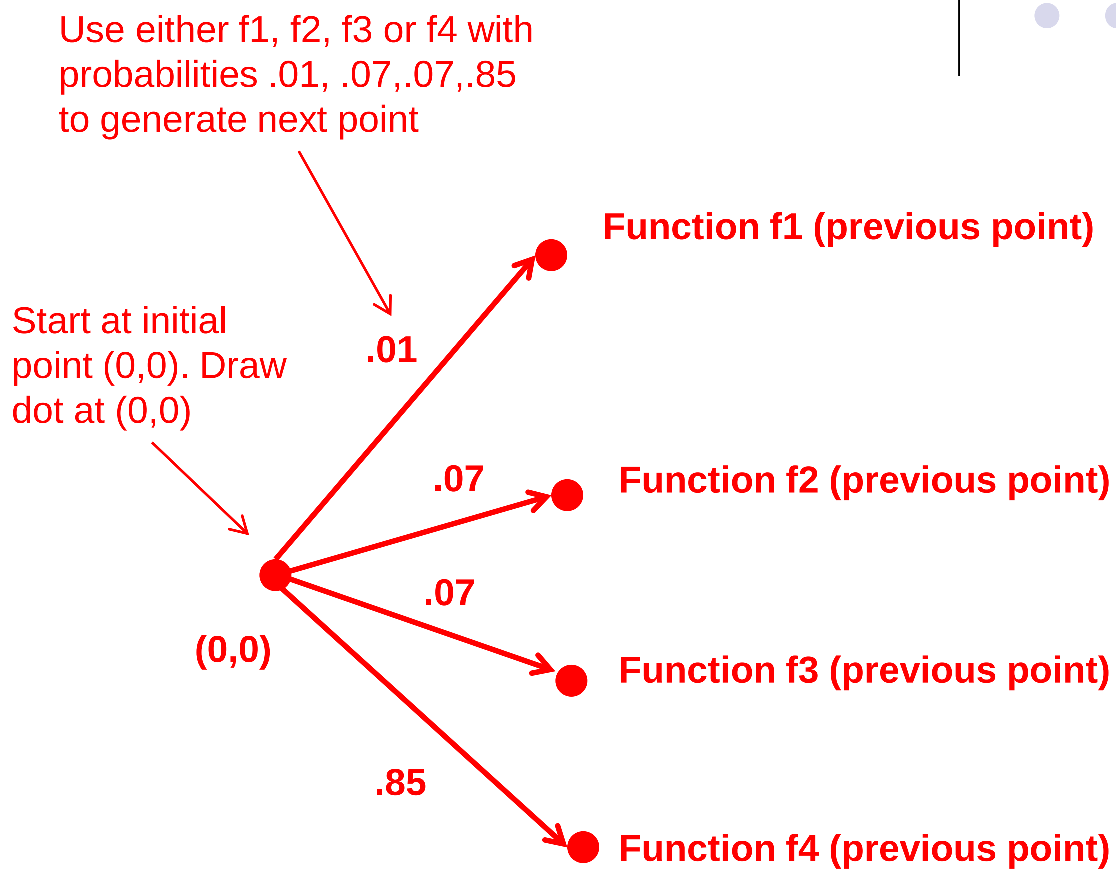
M = 40 L = 3

- A good starting point **p** is (115, 121)

Iterated Function Systems (IFS)

- Recursively call a function
- Does result converge to an image? What image?
- IFS's converge to an image
- Examples:
  - The Fern
  - The Mandelbrot set

The Fern



{Ref: Peitgen: Science of Fractals, p.221 ff} {Barnsley & Sloan, "A Better way to Compress Images" BYTE, Jan 1988, p.215}

The Fern

Each new point (new.x,new.y) is formed from the prior point (old.x,old.y) using the rule:

new.x := a[index] * old.x + c[index] * old.y + tx[index];

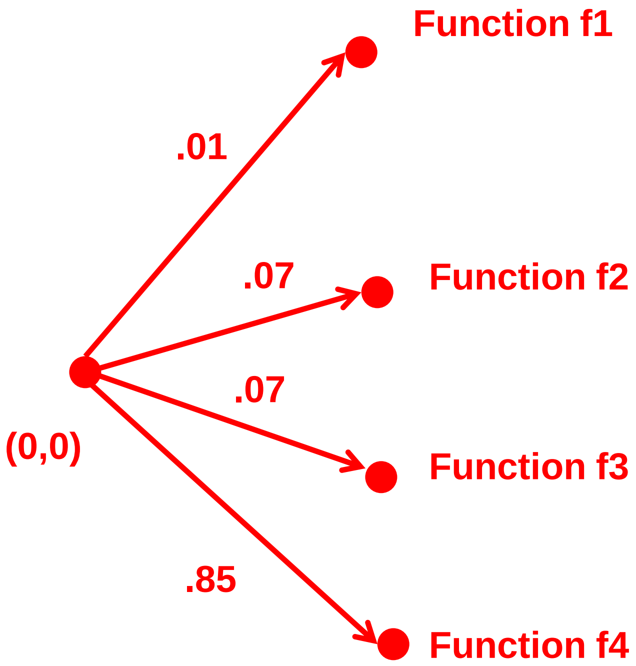
new.y := b[index] * old.x + d[index] * old.y + ty[index];

a[1]:= 0.0; b[1] := 0.0; c[1] := 0.0; d[1] := 0.16;
tx[1] := 0.0; ty[1] := 0.0; (i.e values for function f1)

a[2]:= 0.2; b[2] := 0.23; c[2] :=-0.26; d[2] := 0.22;
tx[2] := 0.0; ty[2] := 1.6; (values for function f2)

a[3]:= -0.15; b[3] := 0.26; c[3] := 0.28; d[3] := 0.24;
tx[3] := 0.0; ty[3] := 0.44; (values for function f3)

a[4]:= 0.85; b[4] := -0.04; c[4] := 0.04; d[4] := 0.85;
tx[4] := 0.0; ty[4] := 1.6; (values for function f4)



Mandelbrot Set

- Based on iteration theory
- Function of interest:

$$f(z) = (s)^2 + c$$

- Sequence of values (or orbit):

$$d_1 = (s)^2 + c$$

$$d_2 = ((s)^2 + c)^2 + c$$

$$d_3 = (((s)^2 + c)^2 + c)^2 + c$$

$$d_4 = ((((s)^2 + c)^2 + c)^2 + c)^2 + c$$

Mandelbrot Set

- Orbit depends on *s* and *c*
- Basic question,:
  - For given *s* and *c,*
    - does function stay finite? (within Mandelbrot set)
    - explode to infinity? (outside Mandelbrot set)
- Definition: if |d| < 1, orbit is finite else inifinite
- Examples orbits:
  - $s = 0$, $c = -1$, orbit = 0,-1,0,-1,0,-1,0,-1,…..*finite*
  - $s = 0$, $c = 1$, orbit = 0,1,2,5,26,677…… *explodes*

Mandelbrot Set

- Mandelbrot set: use complex numbers for *c* and *s*
- Always set $s = 0$
- Choose c as a complex number
- For example:
  - $s = 0$, $c = 0.2 + 0.5i$
- Hence, orbit:
  - $0, \quad c, \quad c^2 + c, \quad (c^2 + c)^2 + c, \ldots\ldots\ldots$
- Definition: Mandelbrot set includes all finite orbit *c*

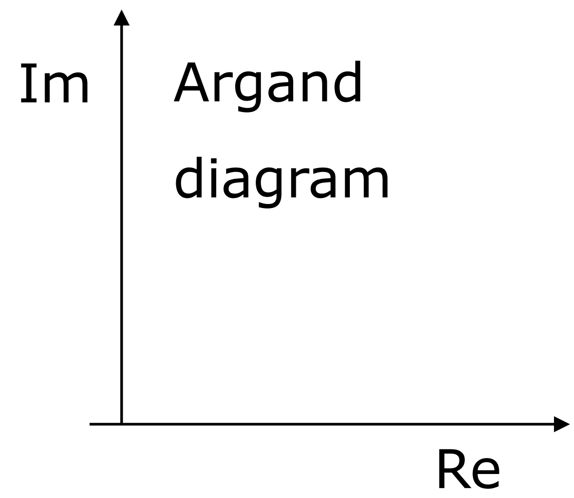
Mandelbrot Set

- Some complex number math:

$$i * i = -1$$

- Example:

$$2i * 3i = -6$$



- Modulus of a complex number, z = ai + b:

$$|z| = \sqrt{a^2 + b^2}$$

- Squaring a complex number:

$$(x + yi)^2 = (x^2 - y^2) + (2xy)i$$

Mandelbrot Set

- Examples: Calculate first 3 terms
  - with s=2, c=-1, terms are

$$2^2 - 1 = 3$$

$$3^2 - 1 = 8$$

$$8^2 - 1 = 63$$

  - with s = 0, c = -2+i

$$(x + yi)^2 = (x^2 - y^2) + (2xy)i$$

$$0 + (-2 + i) = -2 + i$$

$$(-2 + i)^2 + (-2 + i) = 1 - 3i$$

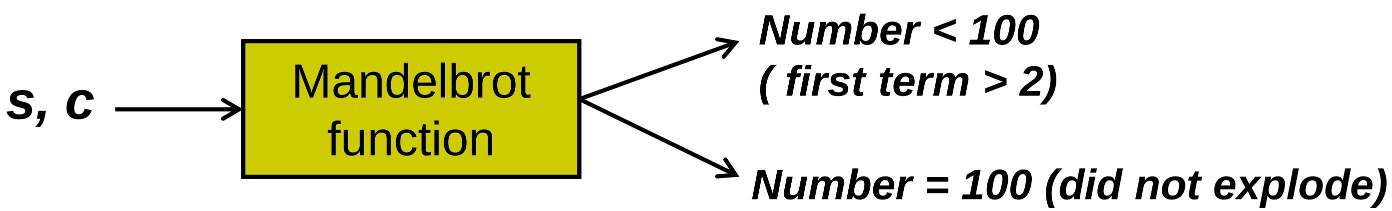
$$(1 - 3i)^2 + (-2 + i) = -10 - 5i$$

Mandelbrot Set

- **Fixed points:** Some complex numbers converge to certain values after *x* iterations.
- **Example:**
  - s = 0, *c* = -0.2 + 0.5i converges to –0.249227 + 0.333677i after 80 iterations
  - **Experiment:** square –0.249227 + 0.333677i and add -0.2 + 0.5i
- Mandelbrot set depends on the fact the convergence of certain complex numbers

Mandelbrot Set Routine

- Math theory says calculate terms to **infinity**
- Cannot iterate forever: our program will hang!
- Instead iterate 100 times
- **Math theorem:**
  - if no term has exceeded 2 after 100 iterations, never will!
- Routine returns:
  - 100, if modulus doesn’t exceed 2 after 100 iterations
  - Number of times iterated before modulus exceeds 2, *or*



Mandelbrot dwell( ) function

$$(x+yi)^2 = (x^2 - y^2) + (2xy)i$$

$$(x+yi)^2 + (c_X + c_Y i) = [(x^2 - y^2) + c_X] + (2xy + c_Y)i$$

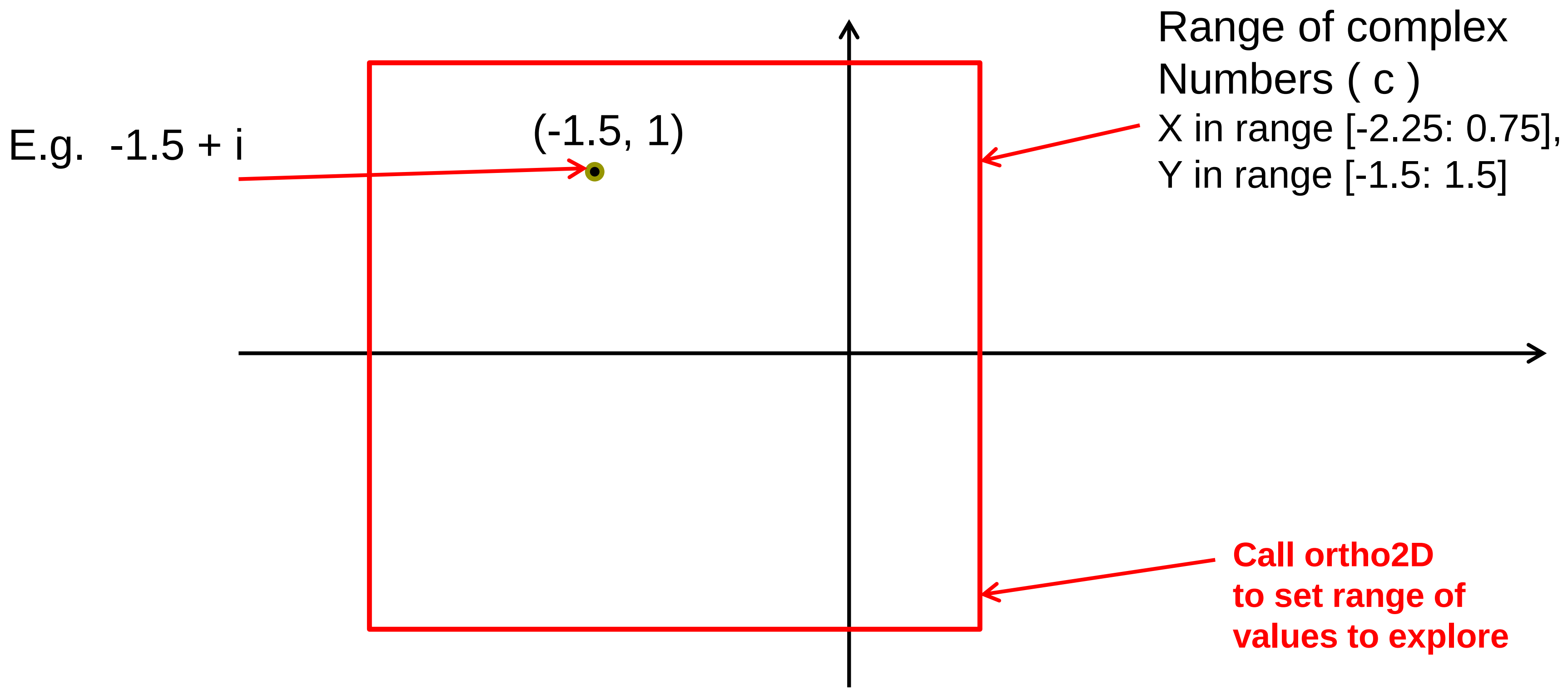
```
int dwell(double cx, double cy)
{ // return true dwell or Num, whichever is smaller
   #define Num 100 // increase this for better pics

   double tmp, dx = cx, dy = cy, fsq = cx*cx + cy*cy;
   for(int count = 0;count <= Num && fsq <= 4; count++)
   {
        tmp = dx;        // save old real part
        dx = dx*dx - dy*dy + cx; // new real part
        dy = 2.0 * tmp * dy + cy; // new imag. Part
        fsq = dx*dx + dy*dy;
   }
   return count; // number of iterations used
}
```

$$[(x^2 - y^2) + c_X]$$

$$(2xy + c_Y)i$$

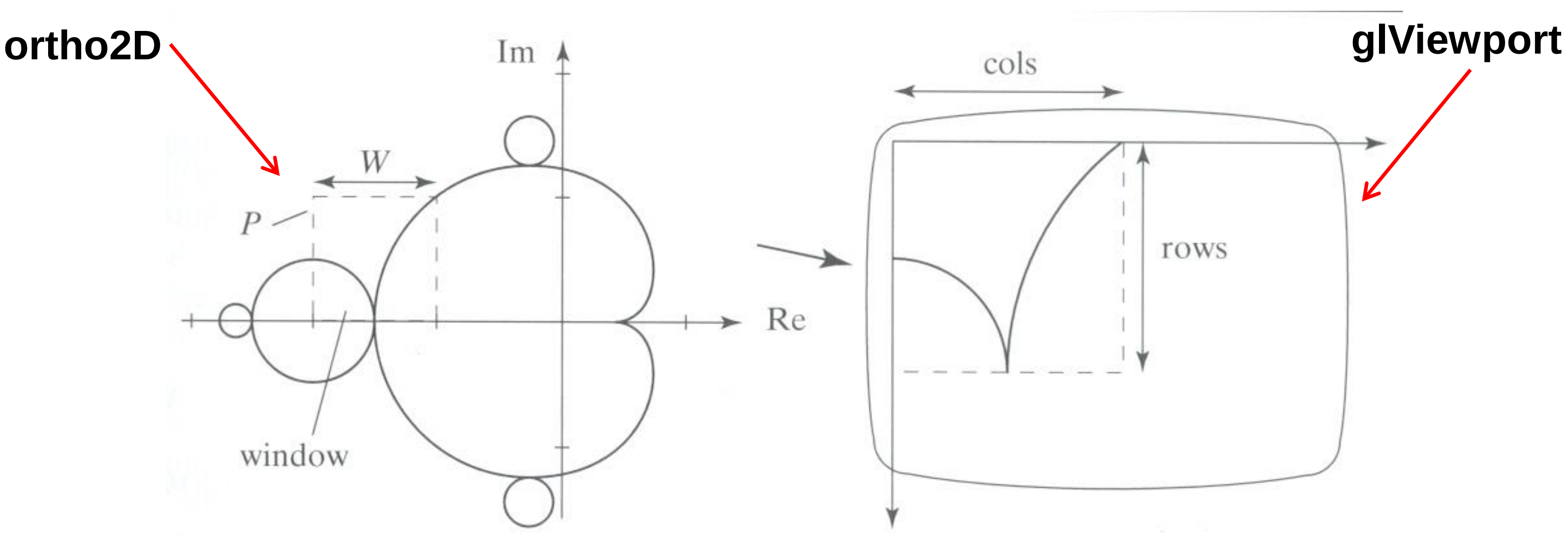
Mandelbrot Set

- Map real part to x-axis
- Map imaginary part to y-axis
- Decide range of complex numbers to investigate. E.g:
  - X in range [-2.25: 0.75], Y in range [-1.5: 1.5]



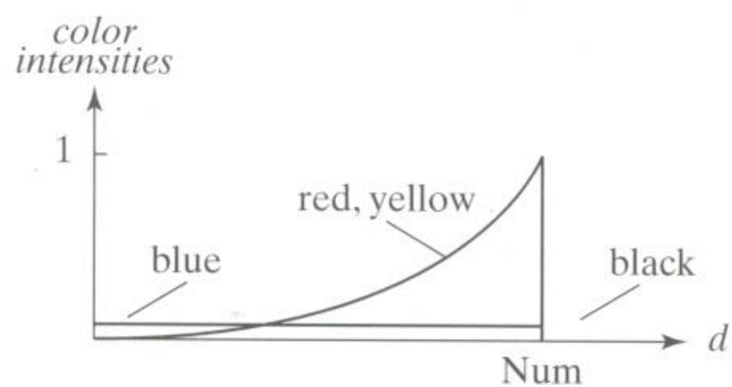
Mandelbrot Set

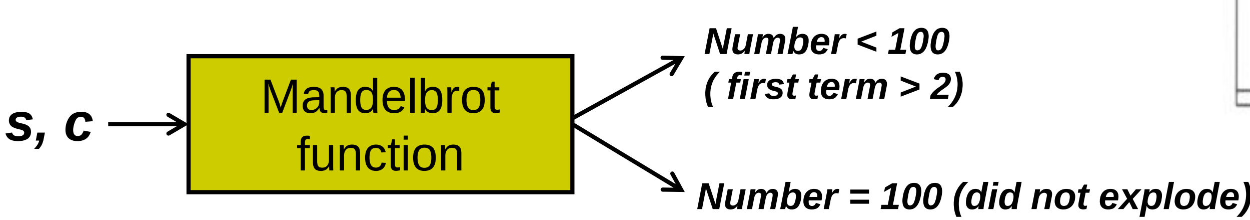
- Set world window (ortho2D) (range of complex numbers to investigate)
  - X in range [-2.25: 0.75], Y in range [-1.5: 1.5]
- Set viewport (glviewport). E.g:
  - Viewport = [V.L, V.R, W, H]= [60,80,380,240]



Mandelbrot Set

- So, for each pixel:
  - For each point (c) in world window call your dwell( ) function
  - Assign color <Red,Green,Blue> based on dwell( ) return value
- Choice of color determines how pretty
- Color assignment:
  - Basic: In set (i.e. dwell( ) = 100), color = black, else color = white
  - Discrete: Ranges of return values map to same color
    - E.g 0 – 20 iterations = color 1
    - 20 – 40 iterations = color 2, etc.
  - Continuous: Use a function





Free Fractal Generating Software

- Fractint
- FracZoom
- 3DFrac

References

- Angel and Shreiner, Interactive Computer Graphics, 6th edition, Chapter 9
- Hill and Kelley, Computer Graphics using OpenGL, 3rd edition, Appendix 4