

Computer Graphics (CS 543)

Lecture 3b: Linear Algebra for Graphics (Points, Scalars, Vectors)

Prof Emmanuel Agu

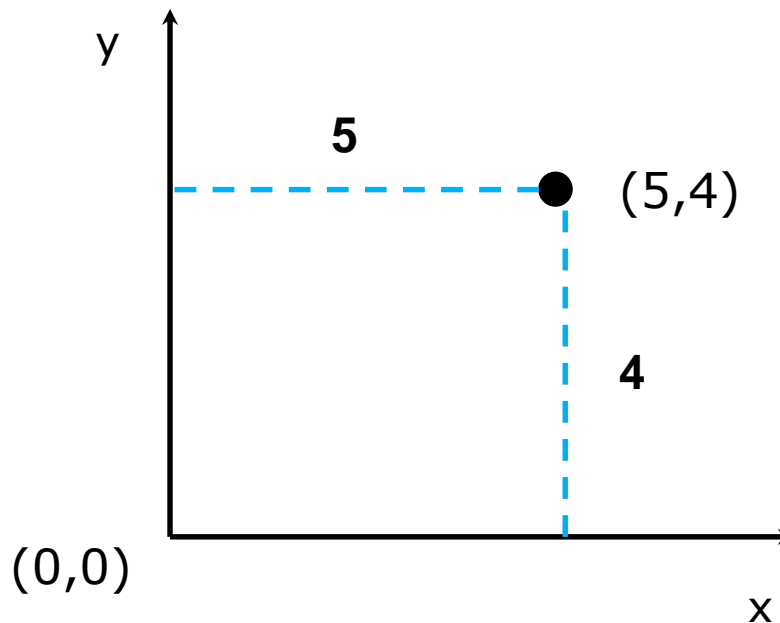
*Computer Science Dept.
Worcester Polytechnic Institute (WPI)*





Points, Scalars and Vectors

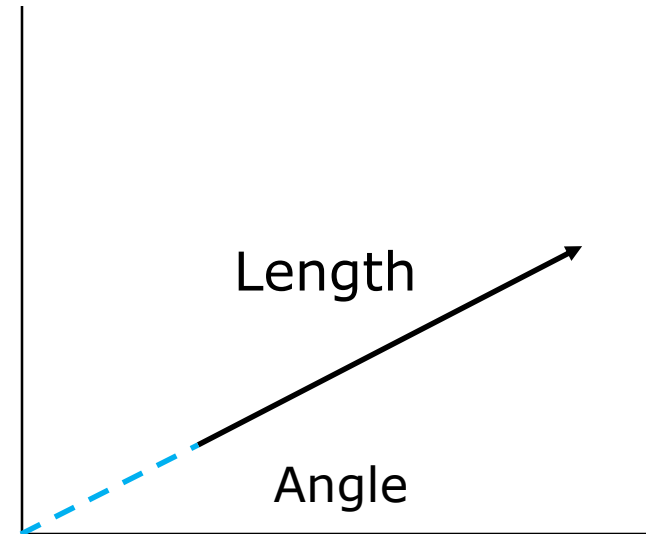
- Points, vectors defined relative to a coordinate system
- **Point:** Location in coordinate system
- Example: Point (5,4)
- Cannot add or scale **points**



Vectors



- Magnitude
- Direction
- **NO** position
- Can be added, scaled, rotated
- CG vectors: 2, 3 or 4 dimensions





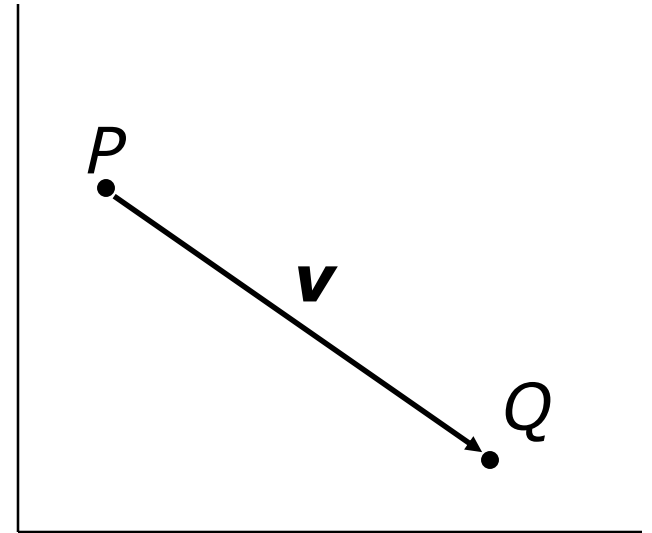
Vector-Point Relationship

- Subtract **2 points** = **vector**

$$\mathbf{v} = Q - P$$

- point + vector = point

$$P + \mathbf{v} = Q$$



Vector Operations



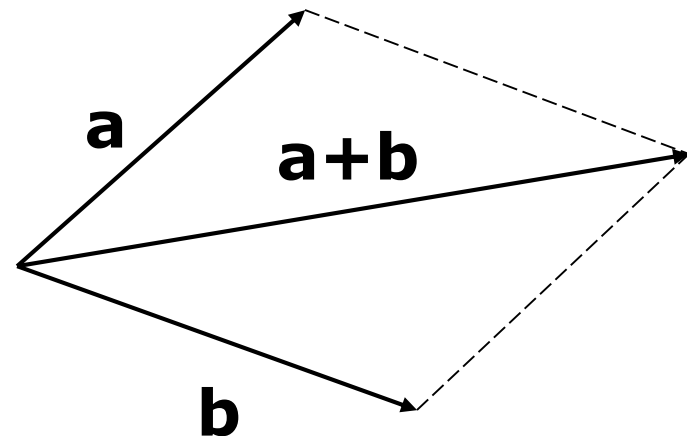
- Define vectors

$$\mathbf{a} = (a_1, a_2, a_3)$$

$$\mathbf{b} = (b_1, b_2, b_3)$$

Then vector addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

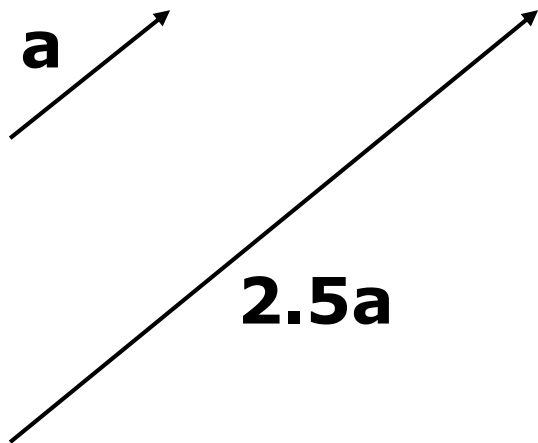




Vector Operations

- Define scalar, s
- Scaling vector by a scalar

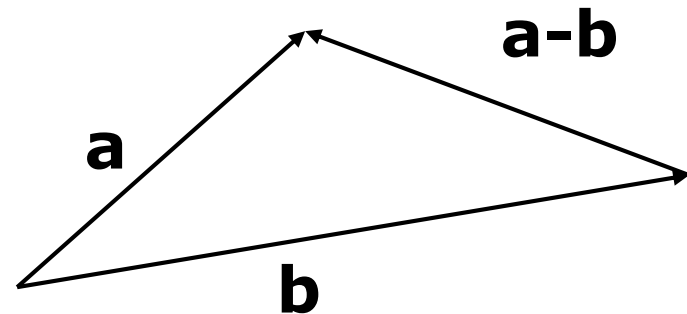
$$\mathbf{as} = (a_1s, a_2s, a_3s)$$



Note vector subtraction:

$$\mathbf{a} - \mathbf{b}$$

$$= (a_1 + (-b_1), a_2 + (-b_2), a_3 + (-b_3))$$





Vector Operations: Examples

- Scaling vector by a scalar
- Vector addition:

$$\mathbf{a}s = (a_1s, a_2s, a_3s)$$

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

- For example, if $\mathbf{a}=(2,5,6)$ and $\mathbf{b}=(-2,7,1)$ and $s=6$, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = (0, 12, 7)$$

$$\mathbf{a}s = (a_1s, a_2s, a_3s) = (12, 30, 36)$$



Affine Combination

- Given a vector

$$\mathbf{a} = (a_1, a_2, a_3, \dots, a_n)$$

$$a_1 + a_2 + \dots + a_n = 1$$

- Affine combination: Sum of all components = 1

- Convex affine = affine + no negative component

i.e

$$a_1, a_2, \dots, a_n = \textit{non-negative}$$



Magnitude of a Vector

- Magnitude of \mathbf{a}

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

- Normalizing a vector (unit vector)

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\text{vector}}{\text{magnitude}}$$

- Note magnitude of normalized vector = 1. i.e

$$\sqrt{a_1^2 + a_2^2 + \dots + a_n^2} = 1$$



Magnitude of a Vector

- Example: if $\mathbf{a} = (2, 5, 6)$

- Magnitude of \mathbf{a} $|\mathbf{a}| = \sqrt{2^2 + 5^2 + 6^2} = \sqrt{65}$

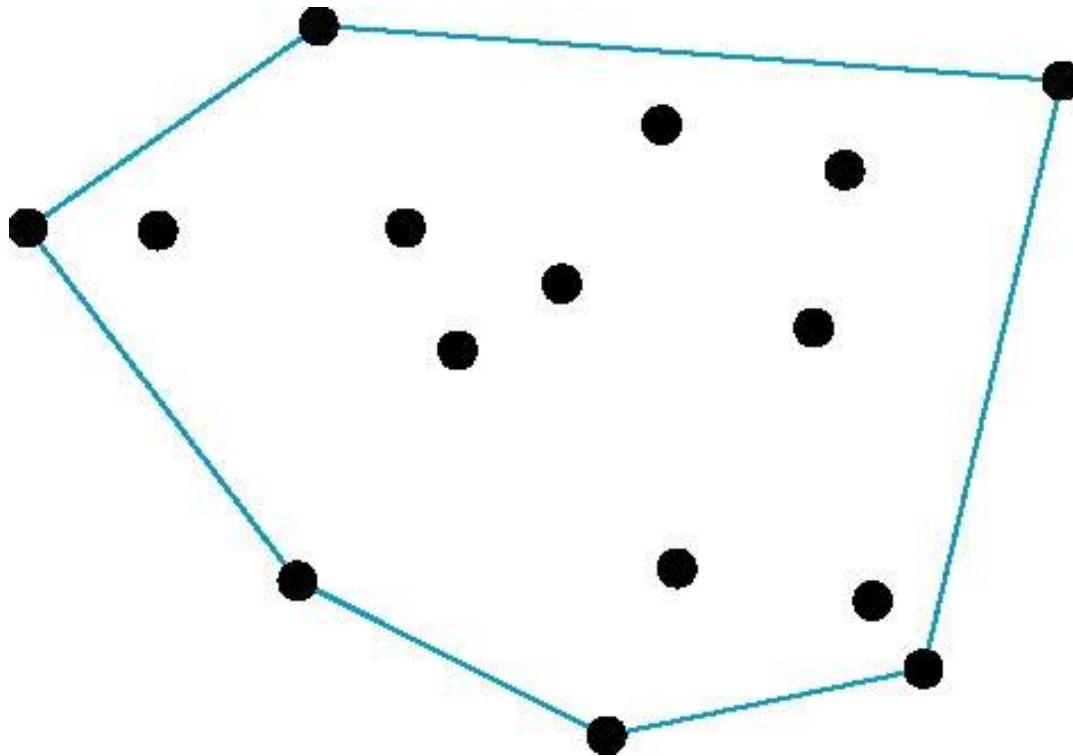
- Normalizing \mathbf{a}

$$\hat{\mathbf{a}} = \left(\frac{2}{\sqrt{65}}, \frac{5}{\sqrt{65}}, \frac{6}{\sqrt{65}} \right)$$



Convex Hull

- Smallest convex object containing P_1, P_2, \dots, P_n
- Formed by “shrink wrapping” points





Dot Product (Scalar product)

- Dot product,

$$d = \mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n$$

- For example, if $a=(2,3,1)$ and $b=(0,4,-1)$

then

$$\begin{aligned} a \cdot b &= (2 \times 0) + (3 \times 4) + (1 \times -1) \\ &= 0 + 12 - 1 = 11 \end{aligned}$$



Properties of Dot Products

- Symmetry (or commutative):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- Linearity:

$$(\mathbf{a} + \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$$

- Homogeneity:

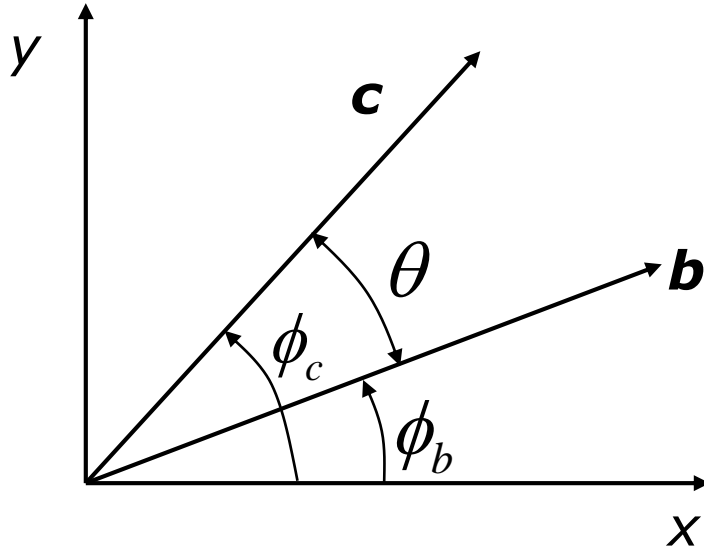
$$(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$$

- And

$$|\mathbf{b}^2| = \mathbf{b} \cdot \mathbf{b}$$



Angle Between Two Vectors

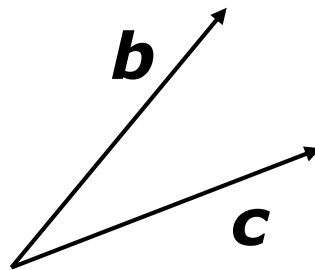


$$\mathbf{b} = (|\mathbf{b}| \cos \phi_b, |\mathbf{b}| \sin \phi_b)$$

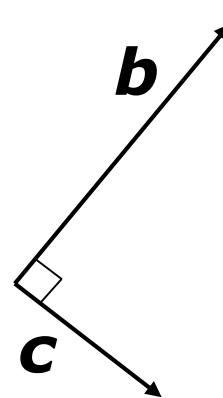
$$\mathbf{c} = (|\mathbf{c}| \cos \phi_c, |\mathbf{c}| \sin \phi_c)$$

$$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}| |\mathbf{c}| \cos \theta$$

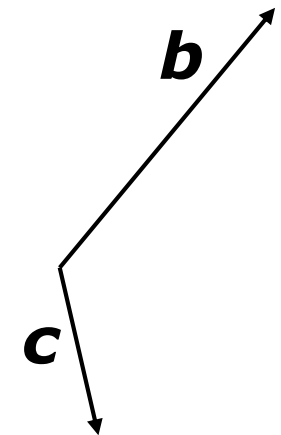
Sign of $\mathbf{b} \cdot \mathbf{c}$:



$$\mathbf{b} \cdot \mathbf{c} > 0$$



$$\mathbf{b} \cdot \mathbf{c} = 0$$



$$\mathbf{b} \cdot \mathbf{c} < 0$$



Angle Between Two Vectors

- **Problem:** Find angle b/w vectors $\mathbf{b} = (3,4)$ and $\mathbf{c} = (5,2)$
- **Step 1:** Find magnitudes of vectors \mathbf{b} and \mathbf{c}

$$|\mathbf{b}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|\mathbf{c}| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

- **Step 2:** Normalize vectors \mathbf{b} and \mathbf{c}

$$\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5} \right)$$

$$\hat{\mathbf{c}} = \left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}} \right)$$



Angle Between Two Vectors

- **Step 3:** Find angle as dot product $\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}$

$$\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} = \left(\frac{3}{5}, \frac{4}{5} \right) \cdot \left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}} \right)$$

$$\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} = \frac{15}{5\sqrt{29}} + \frac{8}{5\sqrt{29}} = \frac{23}{5\sqrt{29}} = 0.85422$$

- **Step 4:** Find angle as inverse cosine

$$\theta = \cos(0.85422) = 31.326^\circ$$

Standard Unit Vectors

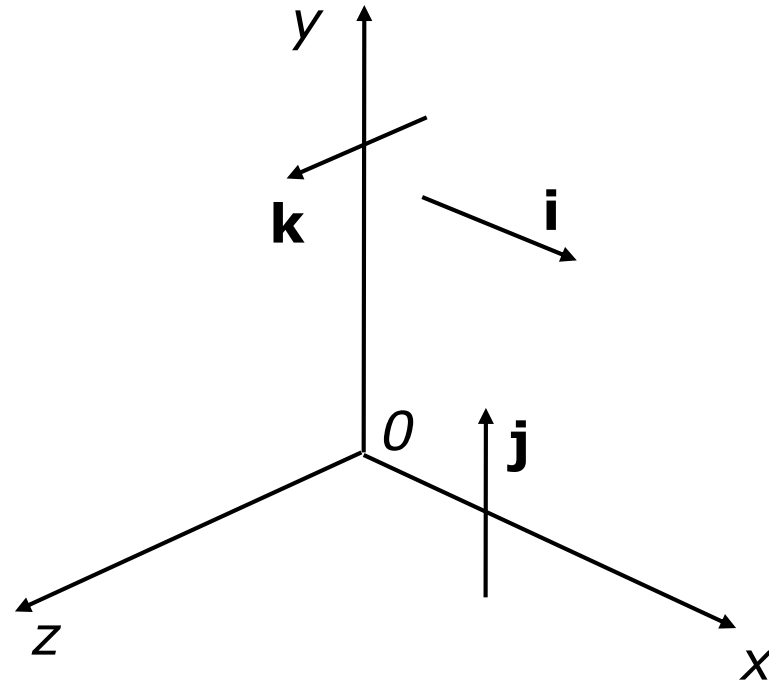


Define

$$\mathbf{i} = (1, 0, 0)$$

$$\mathbf{j} = (0, 1, 0)$$

$$\mathbf{k} = (0, 0, 1)$$



So that any vector,

$$\mathbf{v} = (a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$



Cross Product (Vector product)

If

$$\mathbf{a} = (a_x, a_y, a_z) \quad \mathbf{b} = (b_x, b_y, b_z)$$

Then

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y) \mathbf{i} - (a_x b_z - a_z b_x) \mathbf{j} + (a_x b_y - a_y b_x) \mathbf{k}$$

Remember using determinant

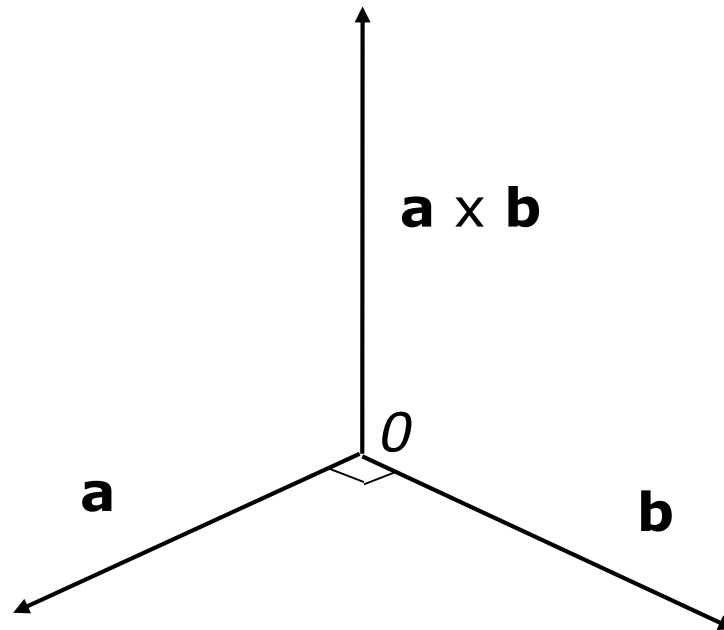
$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Note: $\mathbf{a} \times \mathbf{b}$ is perpendicular to \mathbf{a} and \mathbf{b}

Cross Product



Note: $\mathbf{a} \times \mathbf{b}$ is perpendicular to both \mathbf{a} and \mathbf{b}





Cross Product (Vector product)

Calculate $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = (3,0,2)$ and $\mathbf{b} = (4,1,8)$

$$\mathbf{a} = (3,0,2) \qquad \mathbf{b} = (4,1,8)$$

Using determinant

$$\begin{vmatrix} i & j & k \\ 3 & 0 & 2 \\ 4 & 1 & 8 \end{vmatrix}$$

Then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (0 - 2)\mathbf{i} - (24 - 8)\mathbf{j} + (3 - 0)\mathbf{k} \\ &= -2\mathbf{i} - 16\mathbf{j} + 3\mathbf{k} \end{aligned}$$

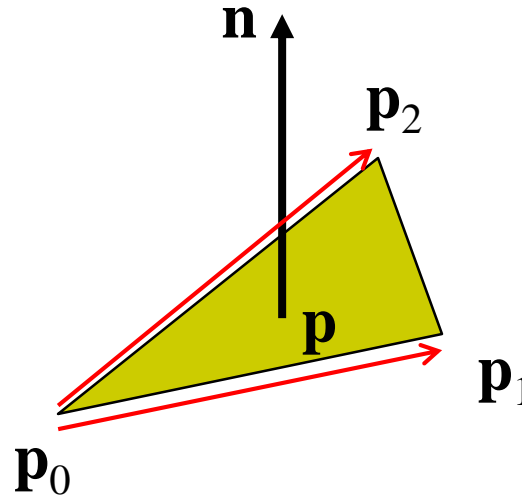
Normal for Triangle using Cross Product Method



plane $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$

$$\mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_0) \times (\mathbf{p}_1 - \mathbf{p}_0)$$

normalize $\mathbf{n} \leftarrow \mathbf{n} / |\mathbf{n}|$

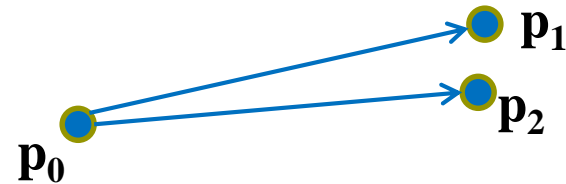


Note that right-hand rule determines outward face



Newell Method for Normal Vectors

- Problems with cross product method:
 - calculation difficult by hand, tedious
 - If 2 vectors almost parallel, cross product is small
 - Numerical inaccuracy may result



- Proposed by Martin Newell at Utah (teapot guy)
 - Uses formulae, suitable for computer
 - Compute during mesh generation
 - Robust!



Newell Method Example

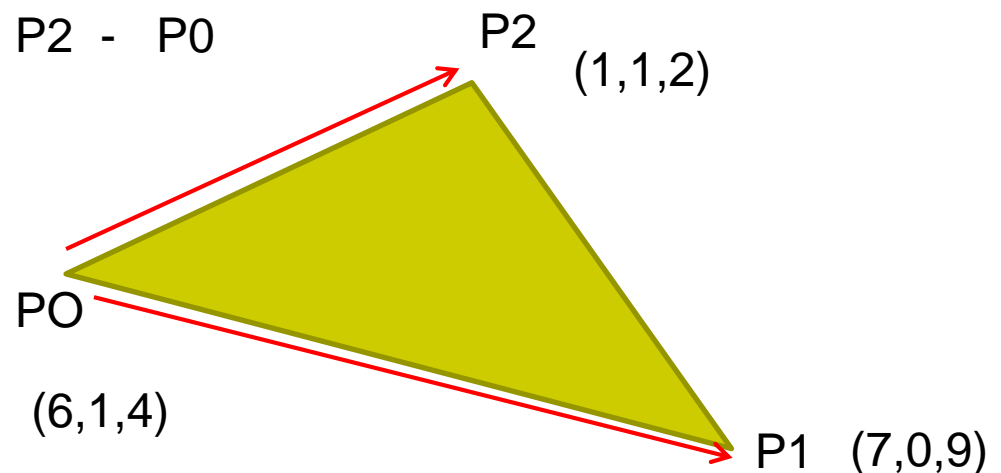
- Example: Find normal of polygon with vertices $P_0 = (6,1,4)$, $P_1=(7,0,9)$ and $P_2 = (1,1,2)$

- Using simple cross product:

$$((7,0,9)-(6,1,4)) \times ((1,1,2)-(6,1,4)) = (2,-23,-5)$$

$P_1 - P_0$

$P_2 - P_0$



Newell Method for Normal Vectors



- Formulae: Normal $N = (m_x, m_y, m_z)$

$$m_x = \sum_{i=0}^{N-1} (y_i - y_{next(i)}) (z_i + z_{next(i)})$$

$$m_y = \sum_{i=0}^{N-1} (z_i - z_{next(i)}) (x_i + x_{next(i)})$$

$$m_z = \sum_{i=0}^{N-1} (x_i - x_{next(i)}) (y_i + y_{next(i)})$$

Newell Method for Normal Vectors



- Calculate x component of normal

$$m_x = \sum_{i=0}^{N-1} (y_i - y_{next(i)}) (z_i + z_{next(i)})$$

$$m_x = (1)(13) + (-1)(11) + (0)(6)$$

$$m_x = 13 - 11 + 0$$

$$m_x = 2$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4

Newell Method for Normal Vectors



- Calculate y component of normal

$$m_y = \sum_{i=0}^{N-1} (z_i - z_{next(i)}) (x_i + x_{next(i)})$$

$$m_y = (-5)(13) + (7)(8) + (-2)(7)$$

$$m_y = -65 + 56 - 14$$

$$m_y = -23$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4



Newell Method for Normal Vectors

- Calculate z component of normal

$$m_z = \sum_{i=0}^{N-1} (x_i - x_{next(i)})(y_i + y_{next(i)})$$

$$m_z = (-1)(1) + (6)(1) + (-5)(2)$$

$$m_z = -1 + 6 - 10$$

$$m_z = -5$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4

Note: Using Newell method yields same result as Cross product method (2,-23,-5)



Finding Vector Reflected From a Surface

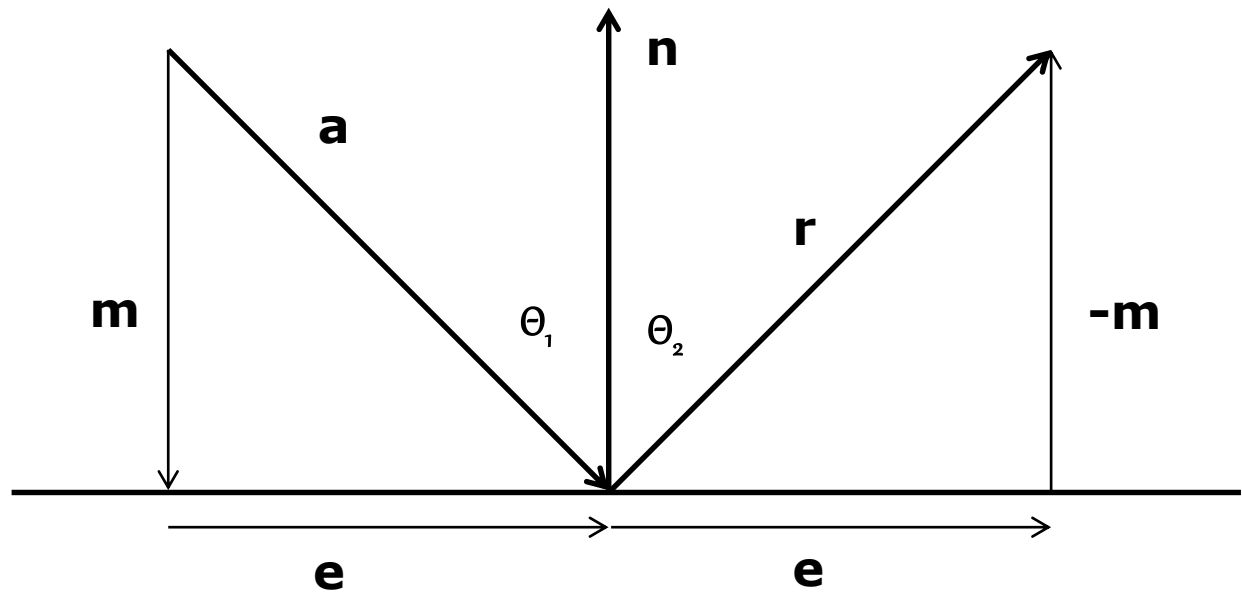
- \mathbf{a} = original vector
- \mathbf{n} = normal vector
- \mathbf{r} = reflected vector
- \mathbf{m} = projection of \mathbf{a} along \mathbf{n}
- \mathbf{e} = projection of \mathbf{a} orthogonal to \mathbf{n}

Note: $\theta_1 = \theta_2$

$$\mathbf{e} = \mathbf{a} - \mathbf{m}$$

$$\mathbf{r} = \mathbf{e} - \mathbf{m}$$

$$\Rightarrow \mathbf{r} = \mathbf{a} - 2\mathbf{m}$$





Forms of Equation of a Line

- Two-dimensional forms of a line

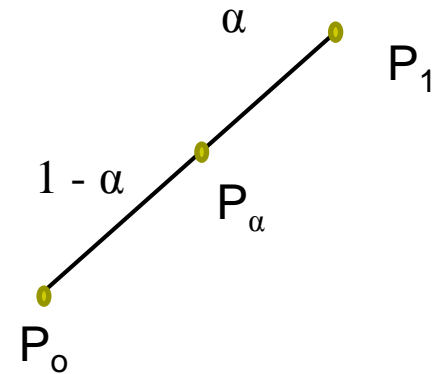
- **Explicit:** $y = mx + h$
- **Implicit:** $ax + by + c = 0$
- **Parametric:**

$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$

$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

- Parametric form of line

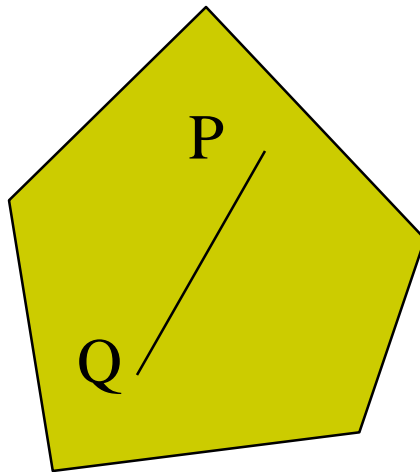
- More robust and general than other forms
- Extends to curves and surfaces



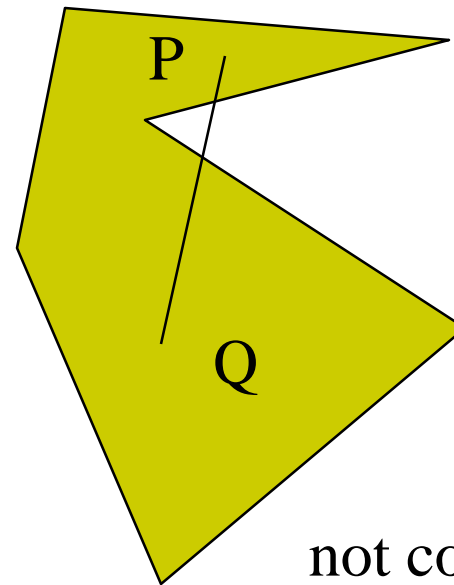


Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



convex



not convex



References

- Angel and Shreiner, Interactive Computer Graphics, 6th edition, Chapter 3
- Hill and Kelley, Computer Graphics using OpenGL, 3rd edition, Sections 4.2 - 4.4