CS 543: Computer Graphics
Lecture 10 (Part III): Curves

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So Far…

- Dealt with straight lines and flat surfaces
- Real world objects include curves
- Need to develop:
  - Representations of curves
  - Tools to render curves

Curve Representation: Explicit

- One variable expressed in terms of another
- Example:

$$z = f(x, y)$$

- Works if one x-value for each y value
- Example: does not work for a sphere

$$z = \sqrt{x^2 + y^2}$$

- Rarely used in CG because of this limitation

Curve Representation: Implicit

- **Algebraic:** represent 2D curve or 3D surface as zeros of a formula
- Example: sphere representation

$$x^2 + y^2 + z^2 - 1 = 0$$

- May restrict classes of functions used
- Polynomial: function which can be expressed as linear combination of integer powers of x, y, z
- Degree of algebraic function: highest sum of powers in function
- Example: yx^4 has degree of 5

Curve Representation: Parametric

- Represent 2D curve as 2 functions, 1 parameter

$$(x(u), y(u))$$

- 3D surface as 3 functions, 2 parameters

$$(x(u,v), y(u,v), z(u,v))$$

- Example: parametric sphere

$$x(q, f) = \cos f \cos q$$
$$y(q, f) = \cos f \sin q$$
$$z(q, f) = \sin f$$

Choosing Representations

- Different representation suitable for different applications
- Implicit representations good for:
  - Computing ray intersection with surface
  - Determining if point is inside/outside a surface
- Parametric representation good for:
  - Breaking surface into small polygonal elements for rendering
  - Subdivide into smaller patches
- Sometimes possible to convert one representation into another

Continuity

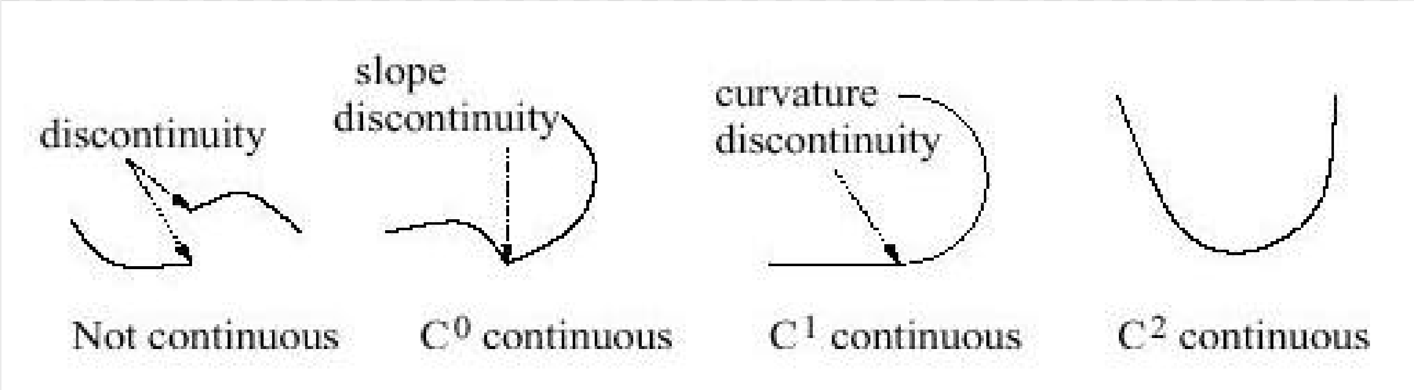
- Consider parametric curve

$$P(u) = (x(u), y(u), z(u))^T$$

- We would like smoothest curves possible
- Mathematically express smoothness as continuity (no jumps)
- **Defn:** if kth derivatives exist, and are continuous, curve has kth order parametric continuity denoted C^k

Continuity

- 0^{th} order means curve is continuous
- 1^{st} order means curve tangent vectors vary continuously, etc
- We generally want highest continuity possible
- However, higher continuity = higher computational cost
- C^2 is usually acceptable

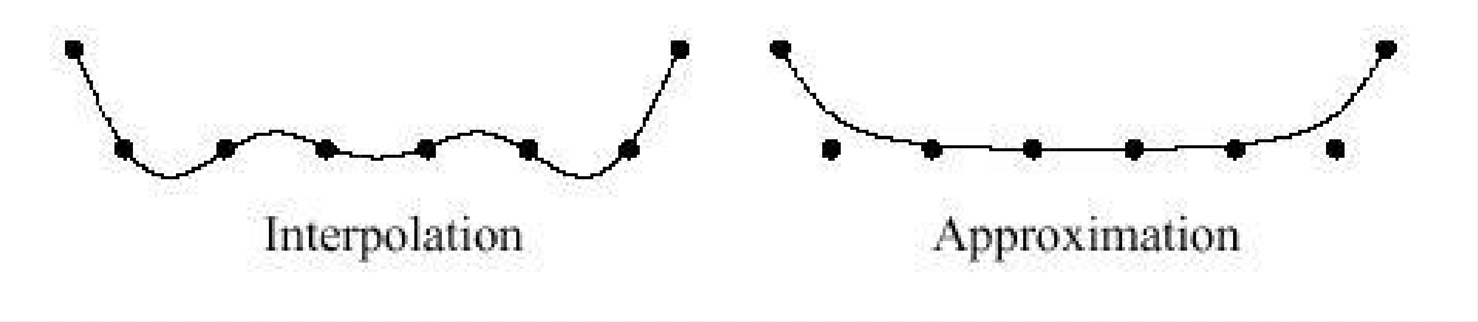


Interactive Curve Design

- Mathematical formula unsuitable for designers
- Prefer to interactively give sequence of control points
- Write procedure:
  - Input: sequence of points
  - Output: parametric representation of curve

Interactive Curve Design

- 1 approach: curves pass through control points (interpolate)
- Example: Lagrangian Interpolating Polynomial
- Difficulty with this approach:
  - Polynomials always have "wiggles"
  - For straight lines wiggling is a problem
- Our approach: merely approximate control points (Bezier, B-Splines)



De Casteljau Algorithm

- Consider smooth curve that approximates sequence of control points [p0,p1,....]

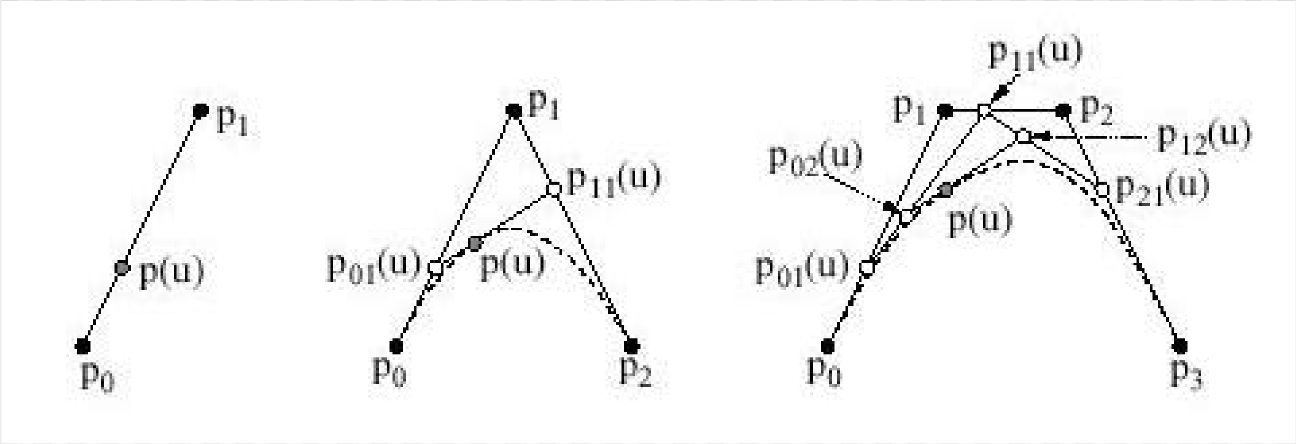
$$p(u) = (1-u)\,p_0 + u p_1 \qquad\qquad 0 \le u \le 1$$

- Blending functions: u and (1 – u) are non-negative and sum to one

De Casteljau Algorithm

- Now consider 3 points
- 2 line segments, P0 to P1 and P1 to P2

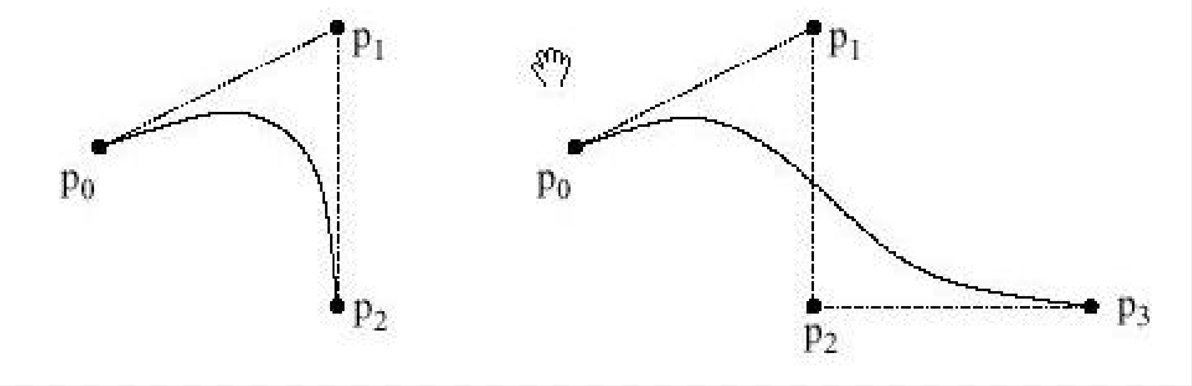
$$p_{01}(u) = (1-u)\,p_0 + u p_1 \qquad\qquad p_{11}(u) = (1-u)\,p_1 + u p_2$$



De Casteljau Algorithm

$$p(u) = (1-u)p_{01} + up_{11}(u)$$
$$= (1-u)^2 p_0 + (2u(1-u))p_1 + u^2 p_2$$

Example: Bezier curves with 3, 4 control points



De Casteljau Algorithm

Blending functions for degree 2 Bezier curve

$$b_{02}(u) = (1-u)^2 \qquad b_{12}(u) = 2u(1-u) \qquad b_{22}(u) = u^2$$

Note: blending functions, non-negative, sum to 1

De Casteljau Algorithm

- Extend to 4 points P0, P1, P2, P3

$$p(u) = (1-u)^3 p_0 + (3u(1-u)^2 p_1 + (3u^2(1-u)) p_2 + u^3$$

- Repeated interpolation is De Casteljau algorithm
- Final result above is Bezier curve of degree 3

De Casteljau Algorithm

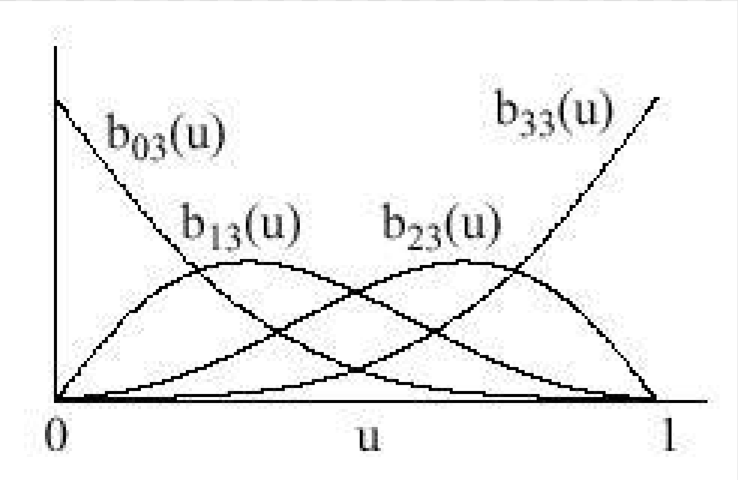
- Blending functions for 4 points
- These polynomial functions called Bernstein's polynomials

$$b_{03}(u) = (1-u)^3$$

$$b_{13}(u) = 3u(1-u)^2$$

$$b_{23}(u) = 3u^2(1-u)$$

$$b_{33}(u) = u^3$$



De Casteljau Algorithm

- Writing coefficient of blending functions gives Pascal's triangle

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

In general, blending function for k Bezier curve has form

$$b_{ik}(u) = \binom{k}{i}(1-u)^{k-i}u^{i} \quad \text{where} \quad \binom{k}{i} = \frac{k!}{i!(k-i)!}$$

De Casteljau Algorithm

- Can express cubic parametric curve in matrix form

$$p(u) = [1, u, u^2, u^3] M_B \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

where

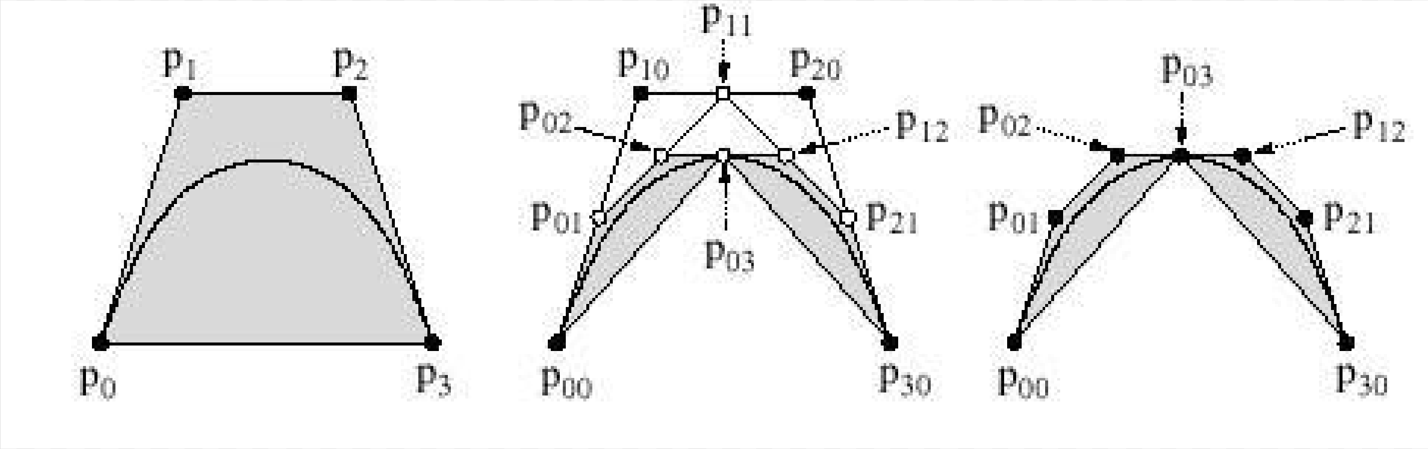
$$M_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Subdividing Bezier Curves

- OpenGL renders flat objects
- To render curves, approximate by small linear segments
- Subdivide curved surface to polygonal patches
- Bezier curves useful for elegant, recursive subdivision
- May have different levels of recursion for different parts of curve or surface
- Example: may subdivide visible surfaces more than hidden surfaces

Subdividing Bezier Curves

- Let (P0… P3) denote original sequence of control points
- Relabel these points as (P00…. P30)
- Repeat interpolation (u = ½) and label vertices as below
- Sequences (P00,P01,P02,P03) and (P03,P12,P21,30) define Bezier curves also
- Bezier Curves can either be straightened or curved recursively in this way



Bezier Surfaces

- Bezier surfaces: interpolate in two dimensions
- This called Bilinear interpolation
- Example: 4 control points, P00, P01, P10, P11, 2 parameters u and v
- Interpolate between
  - P00 and P01 using u
  - P10 and P11 using u
  - Repeat two steps above using v

$$p(u,v) = (1-v)((1-u)p_{00} + up_{01}) + v((1-u)p_{10} + up_{11})$$

$$= (1-v)(1-u)p_{00} + (1-v)up_{01} + v(1-u)p_{10} + vup_{11}$$

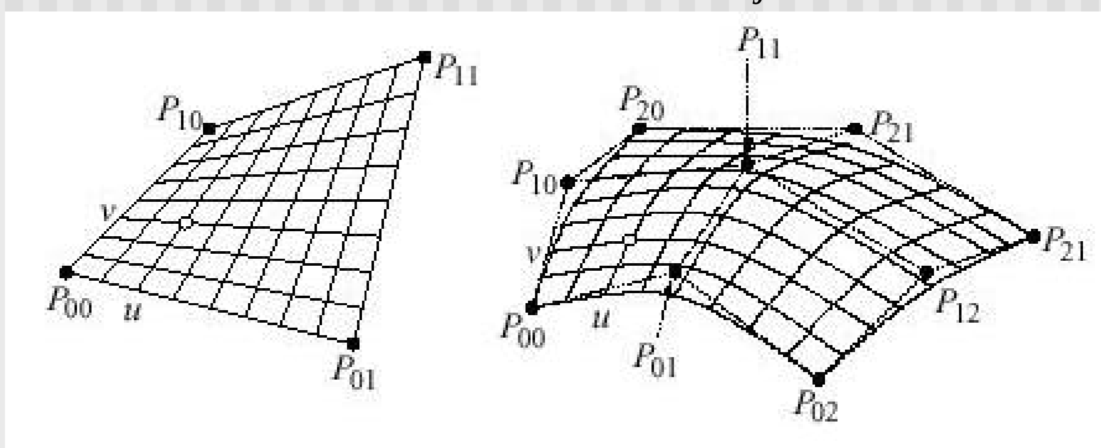
Bezier Surfaces

- Recalling, (1-u) and u are first-degree Bezier blending functions b0,1(u) and b1,1(u)

$$p(u,v) = b_{01}(v)b_{01}(u)\,p_{00} + b_{01}(v)b_{11}b_{01}(u)\,p_{01} + b_{11}(v)b_{11}(u)\,p_{11}$$

Generalizing for cubic

$$p(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_{i,3}(v)b_{j,3}(u)\,p_{i,j}$$

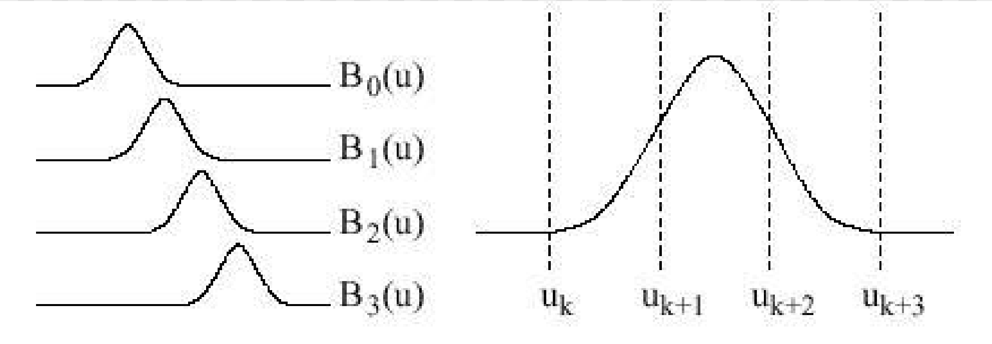


Rendering Bezier patches in openGL: v=u = 1/2

B-Splines

- Bezier curves are elegant but too many control points
- Smoother = more control points = higher order polynomial
- Undesirable: every control point contributes to all parts of curve
- B-splines designed to address Bezier shortcomings
- Smooth blending functions, each non-zero over small range
- Use different polynomial in each range, (**piecewise polynomial**)

$$p(u) = \sum_{i=0}^{m} B_i(u)\, p_i$$



B-spline blending functions, order 2

NURBS

- Encompasses both Bezier curves/surfaces and B-splines
- Non-uniform Rational B-splines (NURBS)
- Rational function is ratio of two polynomials
- NURBS use rational blending functions
- Some curves can be expressed as rational functions but not as simple polynomials
- No known exact polynomial for circle
- Rational parametrization of unit circle on xy-plane:

$$x(u) = \frac{1-u^2}{1+u^2}$$

$$y(u) = \frac{2u}{1+u^2}$$

$$z(u) = 0$$

NURBS

- We can apply homogeneous coordinates to bring in *w*

$$x(u) = 1 - u^2$$
$$y(u) = 2u$$
$$z(u) = 0$$
$$w(u) = 1 + u^2$$

- Using w, we cleanly integrate rational parametrization
- Useful property of NURBS: preserved under transformation
- Thus, we can project control points and then render NURBS

References

- Hill, chapter 11