CS 4731/543: Computer Graphics
Lecture 2 (Part III): Points, Scalars and Vectors

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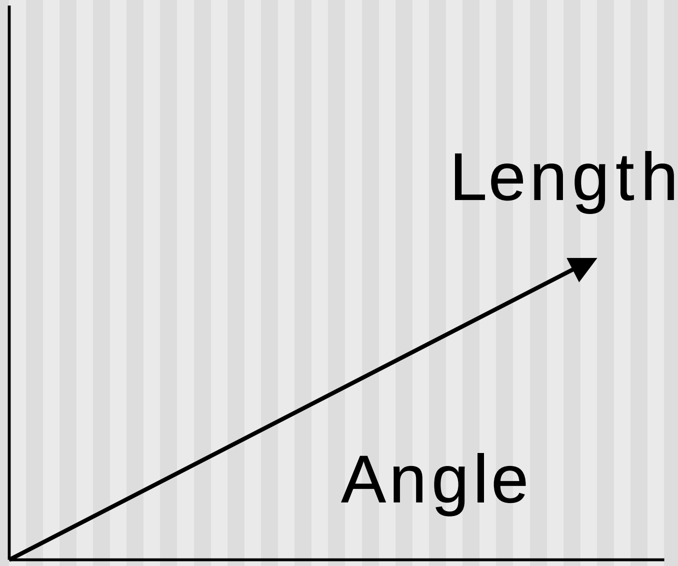
Points, Scalars and Vectors

- Points, vectors defined relative to a coordinate system

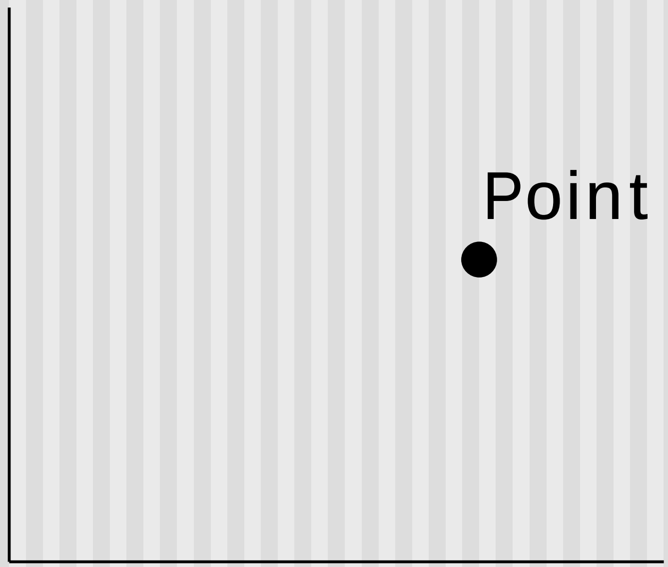
Vectors

- Magnitude
- Direction
- **NO** position
- Can be added, scaled, rotated
- CG vectors: 2, 3 or 4 dimensions



Points

- Location in coordinate system
- Cannot add or scale
- Subtract 2 points = vector



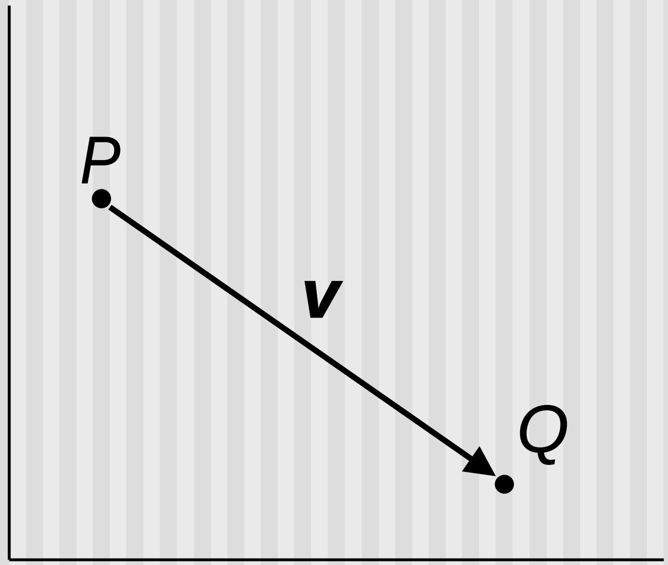
Vector-Point Relationship

- Diff. b/w 2 points = vector

$$\mathbf{v} = Q - P$$

- Sum of point and vector = point

$$\mathbf{v} + P = Q$$



Vector Operations

- Define vectors

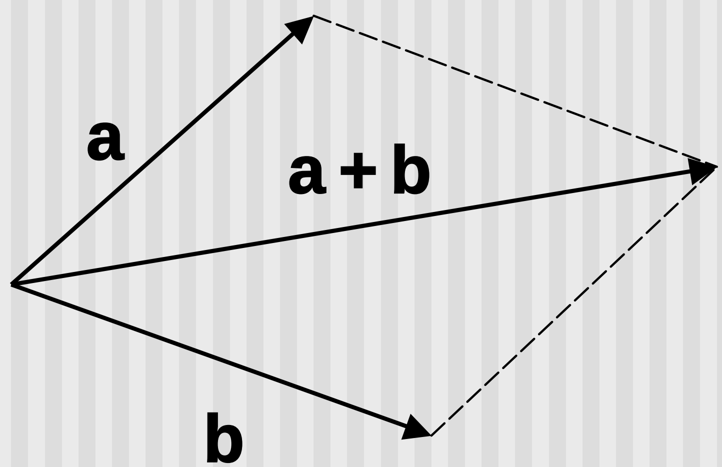
$$\mathbf{a} = (a_{1,} a_2, a_3)$$

$$\mathbf{b} = (b_{1,} b_2, b_3)$$

- and scalar, *s*

Then vector addition:

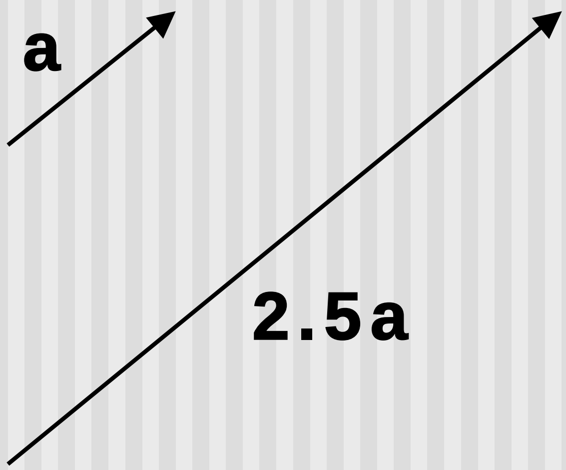
$$\mathbf{a} + \mathbf{b} = (a_1 + b_{1,} a_2 + b_2, a_3 + b_3)$$



Vector Operations

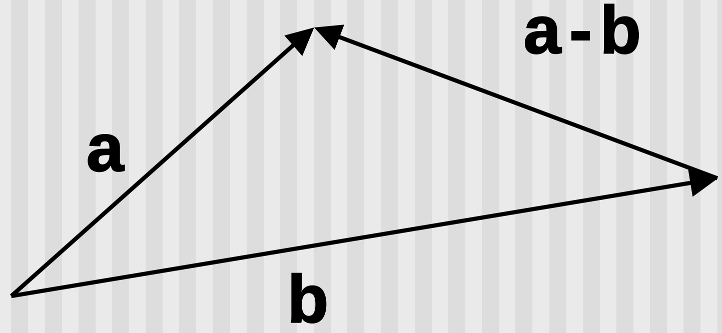
- Scaling vector by a scalar

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s)$$



Note vector subtraction:

$$\mathbf{a} - \mathbf{b}$$
$$= (a_1 + (-b_1), a_2 + (-b_2), a_3 + (-b_3))$$



Vector Operations: Examples

- Scaling vector by a scalar

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s)$$

- Vector addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_{1,} a_2 + b_2, a_3 + b_3)$$

- For example, if **a**=(2,5,6) and **b**=(-2,7,1) and *s*=6, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_{1,} a_2 + b_2, a_3 + b_3) = (0,12,7)$$

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s) = (12,30,36)$$

Affine Combination

- Summation of all components = 1

$$a_1 + a_2 + ...a_n = 1$$

- Convex affine = affine + no negative component

$$a_1, a_2, ...a_n = non-negative$$

Magnitude of a Vector

- Magnitude of **a**

$$|\mathbf{a}| = \sqrt{a_1{}^2 + a_2{}^2 \ldots\ldots\ldots + a_n{}^2}$$

- Normalizing a vector (unit vector)

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{vector}{magnitude}$$

- Note magnitude of normalized vector = 1. i.e

$$\sqrt{a_1{}^2 + a_2{}^2 \ldots\ldots\ldots + a_n{}^2} = 1$$

Dot Product (Scalar product)

- Dot product,

$$d = \mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 \ldots\ldots\ldots + a_3 \cdot b_3$$

- For example, if *a*=(2,3,1) and *b*=(0,4,-1) then

$$a \cdot b = 2 \cdot 0 + 3 \cdot 4 + 1 \cdot -1$$

$$= 0 + 12 - 1 = 11$$

Dot Product

- Try your hands at these:
  - $(2, 2, 2, 2) \bullet (4, 1, 2, 1.1)$
  - $(2, 3, 1) \bullet (0, 4, -1)$

Dot Product

- Try your hands at these:
  - $(2, 2, 2, 2) \bullet (4, 1, 2, 1.1) = 8 + 2 + 4 + 2.2 = 16.2$
  - $(2, 3, 1) \bullet (0, 4, -1) = 0 + 12 - 1 = 11$

Properties of Dot Products

- Symmetry (or commutative):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- Linearity:

$$(\mathbf{a} + \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$$

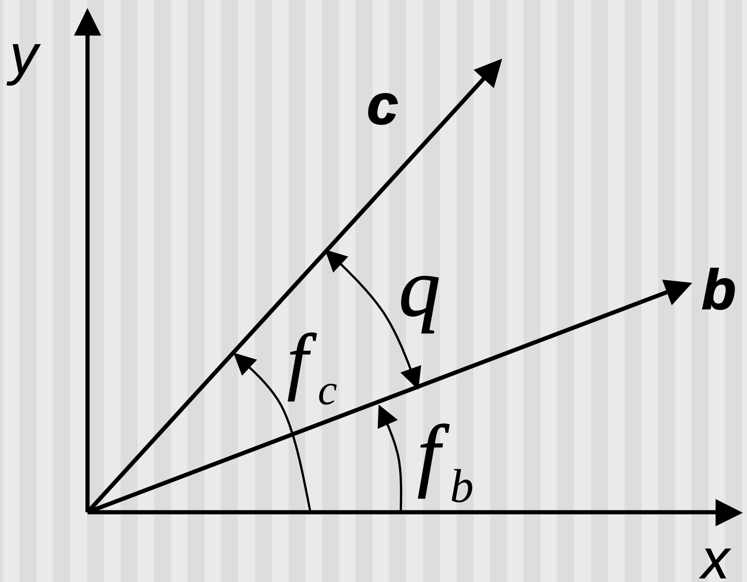
- Homogeneity:

$$(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$$

- And

$$\left|\mathbf{b}^2\right| = \mathbf{b} \cdot \mathbf{b}$$

Angle Between Two Vectors

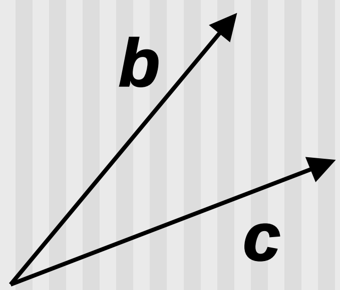


$$\mathbf{b} = \left(|\mathbf{b}|\cos f_b, |\mathbf{b}|\sin f_b\right)$$

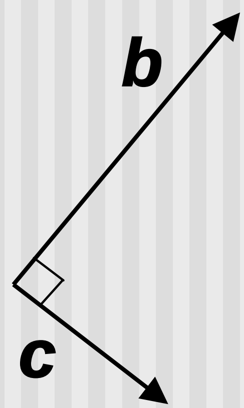
$$\mathbf{c} = \left(|\mathbf{c}|\cos f_c, |\mathbf{c}|\sin f_c\right)$$

$$\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}||\mathbf{c}|\cos q$$

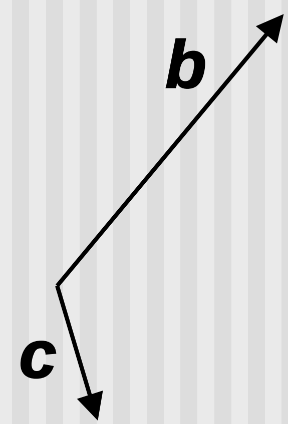
Sign of **b.c**:



b.c > 0



b.c = 0



b.c < 0

Angle Between Two Vectors

- Find the angle b/w the vectors **b** = (3,4) and **c =** (5,2)

Angle Between Two Vectors

- Find the angle b/w the vectors **b** = (3,4) and **c =** (5,2)
  - |**b**|= 5, |**c**|= 5.385

$$\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad \hat{\mathbf{b}} \bullet \hat{\mathbf{c}} = 0.85422 = \cos q$$

$$q = 31.326^{\circ}$$

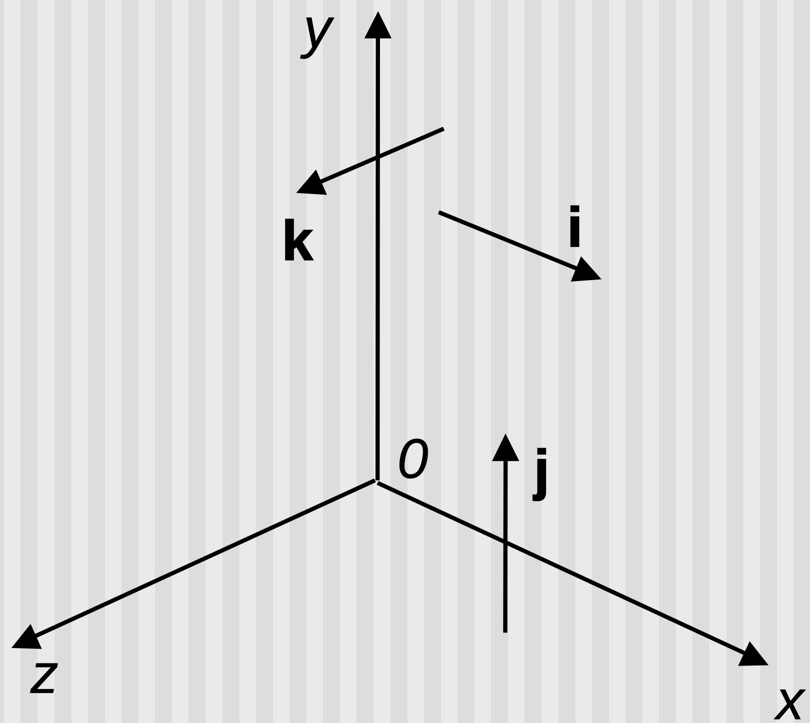
Standard Unit Vectors

Define

$$\mathbf{i} = (1,0,0)$$

$$\mathbf{j} = (0,1,0)$$

$$\mathbf{k} = (0,0,1)$$



So that any vector,

$$\mathbf{v} = (a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Cross Product (Vector product)

If

$$\mathbf{a} = \left(a_x, a_y, a_z\right) \qquad \mathbf{b} = \left(b_x, b_y, b_z\right)$$

Then

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} - (a_x b_z - a_z b_x)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

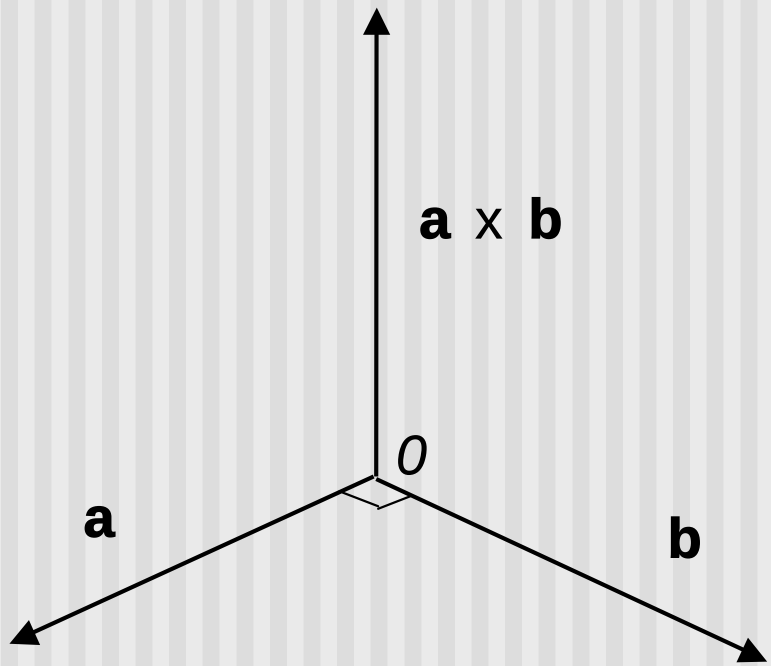
Remember using determinant

$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Note: a x **b** is perpendicular to **a** and **b**

Cross Product

Note: a x **b** is perpendicular to both **a** and **b**



Cross Product

Calculate **a x b** if a = (3,0,2) and **b** = (4,1,8)

Cross Product

Calculate **a x b** if a = (3,0,2) and **b** = (4,1,8)

a x b = -2**i** – 16**j** + 3**k**

References

- Hill, chapter 4.2 - 4.4