CS 543: Computer Graphics

Transformations

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(with lots of help from Prof. Emmanuel Agu :-)

Introduction to Transformations

- A *transformation* changes an object's
  - Size (scaling)
  - Position (translation)
  - Orientation (rotation)
  - Shape (shear)
- We will introduce first in 2D or *(x,y)*, build intuition
- Later, talk about 3D and 4D?
- Transform object by applying sequence of matrix multiplications to object vertices

Why Matrices?

- All transformations can be performed using matrix/vector multiplication
- Allows pre-multiplication of all matrices
- Note: point (x, y) needs to be represented as (x, y, 1), also called ***homogeneous coordinates***

Point Representation

- We use a column matrix (2x1 matrix) to represent a 2D point

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

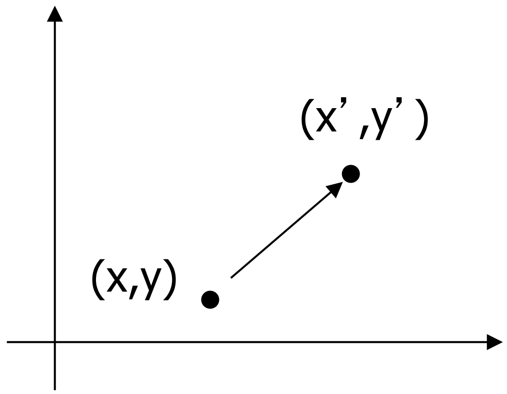
- General form of transformation of a point *(x, y)* to *(x', y')* can be written as:

$$\begin{aligned} x' &= ax + by + c \\ y' &= dx + ey + f \end{aligned} \quad \text{or} \quad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Translation

- To reposition a point along a straight line
- Given point *(x, y)* and translation distance (t_x, t_y)
- The new point: *(x', y')*

$$x' = x + t_x$$
$$y' = y + t_y$$



or

$$P' = P + T \quad \text{where} \quad P' = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad P = \begin{pmatrix} x \\ y \end{pmatrix} \quad T = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

3x3 2D Translation Matrix

$$x' = x + t_x$$
$$y' = y + t_y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

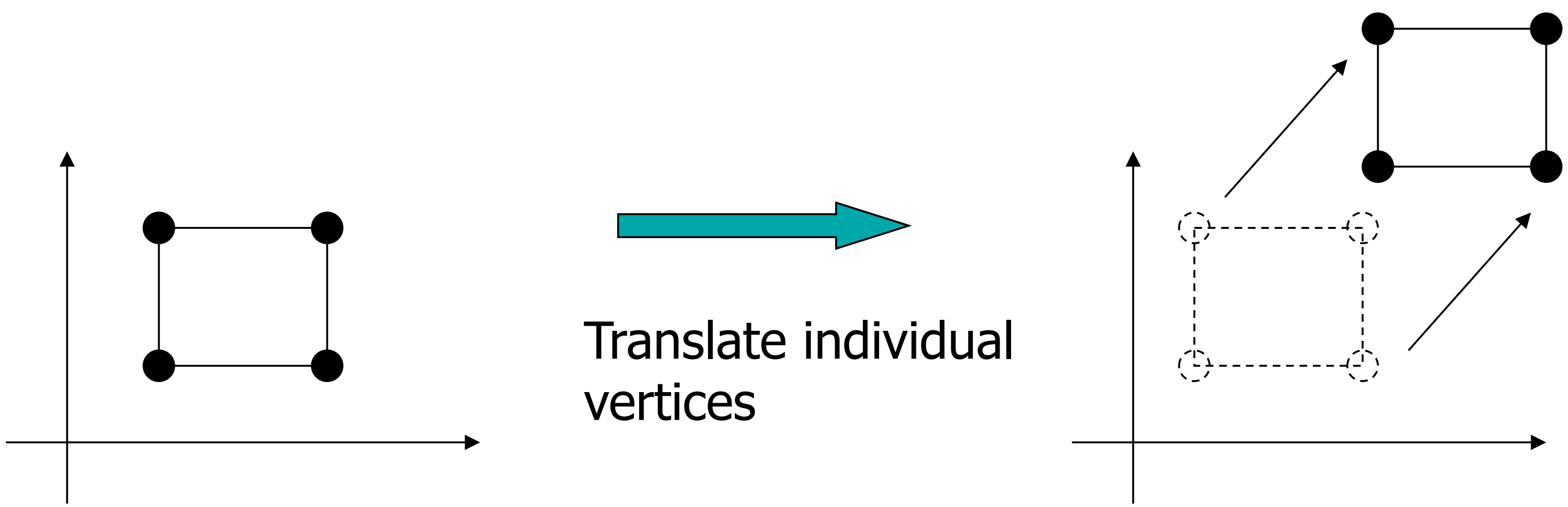
$\Downarrow$ use 3x1 vector

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Note: it becomes a matrix-vector multiplication

Translation of Objects

- How to translate an object with multiple vertices?

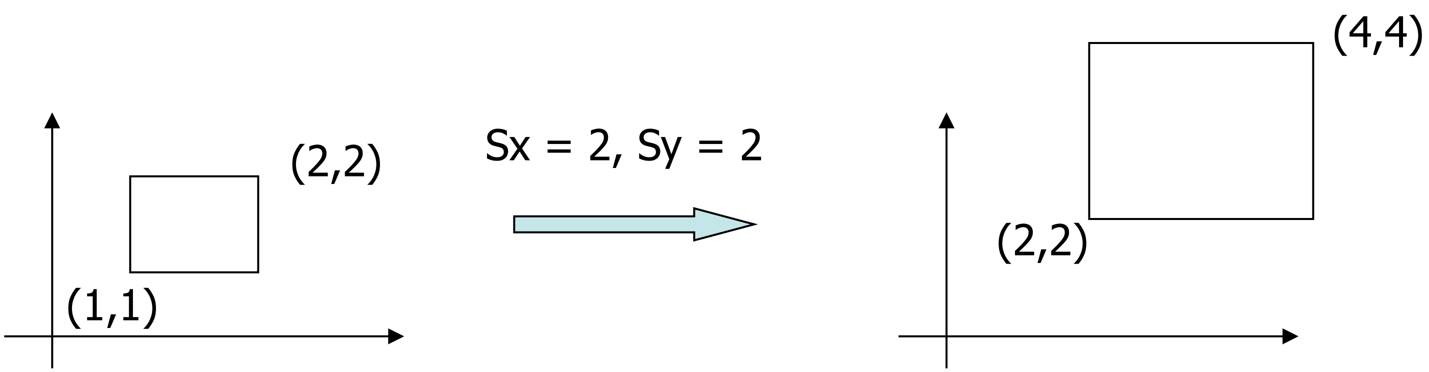


2D Scaling

- Scale: Alter object size by scaling factor (sx, sy). *i.e.*,

$$x' = x * Sx$$
$$y' = y * Sy$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Sx & 0 \\ 0 & Sy \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



3x3 2D Scaling Matrix

*x' = x * Sx*
*y' = y * Sy*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Sx & 0 \\ 0 & Sy \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Downarrow$$

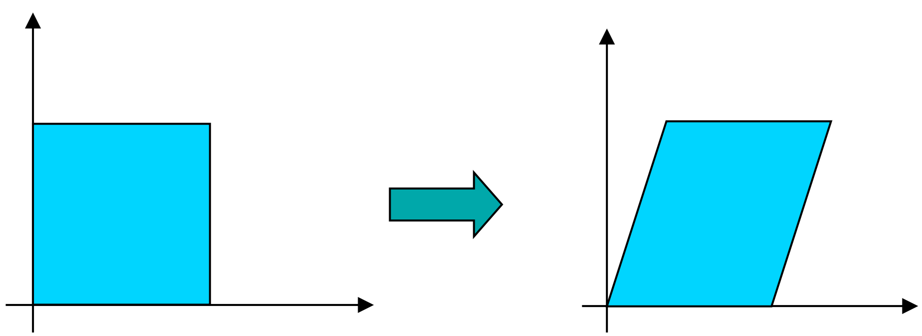
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Shearing

- Y coordinates are unaffected, but x coordinates are translated linearly with y
- That is

$$x' = x + y*h$$
$$y' = y$$

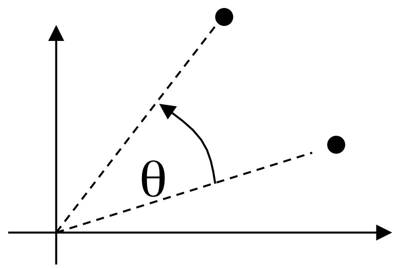
  - h is fraction of *y* to be added to *x*



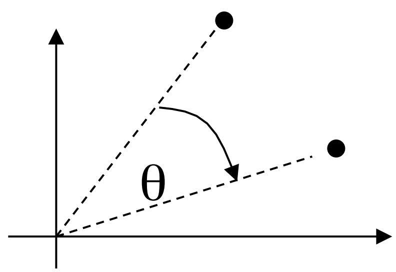
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Rotation

- Default rotation center is origin *(0,0)*



$\theta > 0$: Rotate counter clockwise

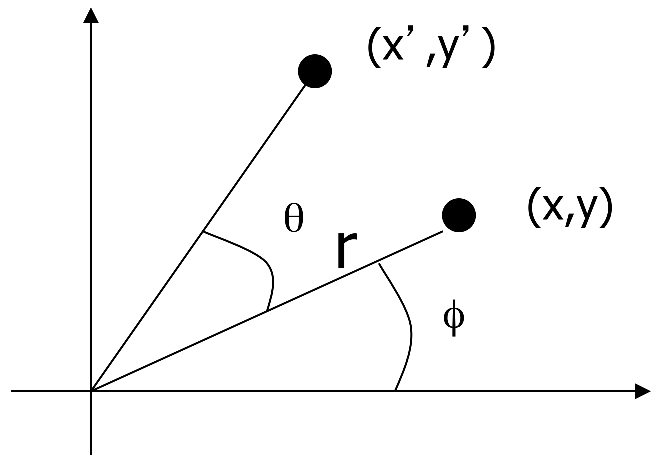


$\theta < 0$: Rotate clockwise

2D Rotation (cont.)

(x,y) -> Rotate *about the origin* by θ

→ (x', y')



How to compute (x', y') ?

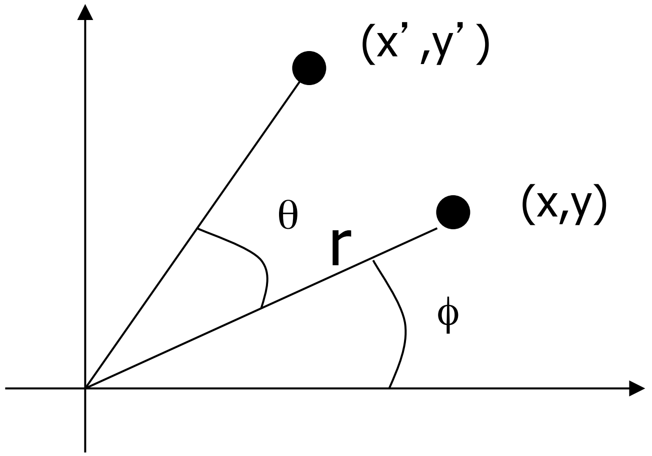
$$x = r*\cos(\phi) \qquad x' = r*\cos(\phi + \theta)$$
$$y = r*\sin(\phi) \qquad y' = r*\sin(\phi + \theta)$$

2D Rotation (cont.)

- Using trig. identities

$$\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi$$

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi$$



x’ = x cos(θ) – y sin(θ)
y’ = x sin(θ) + y cos(θ)

Matrix form?

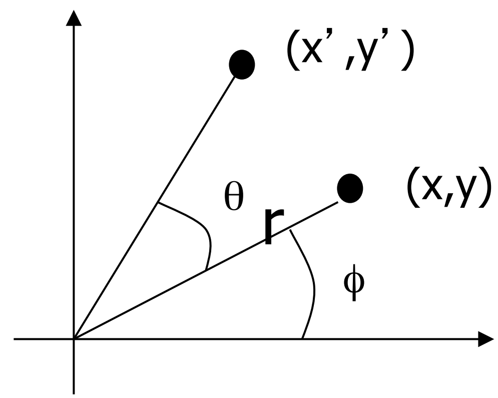
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

3x3 2D Rotation Matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

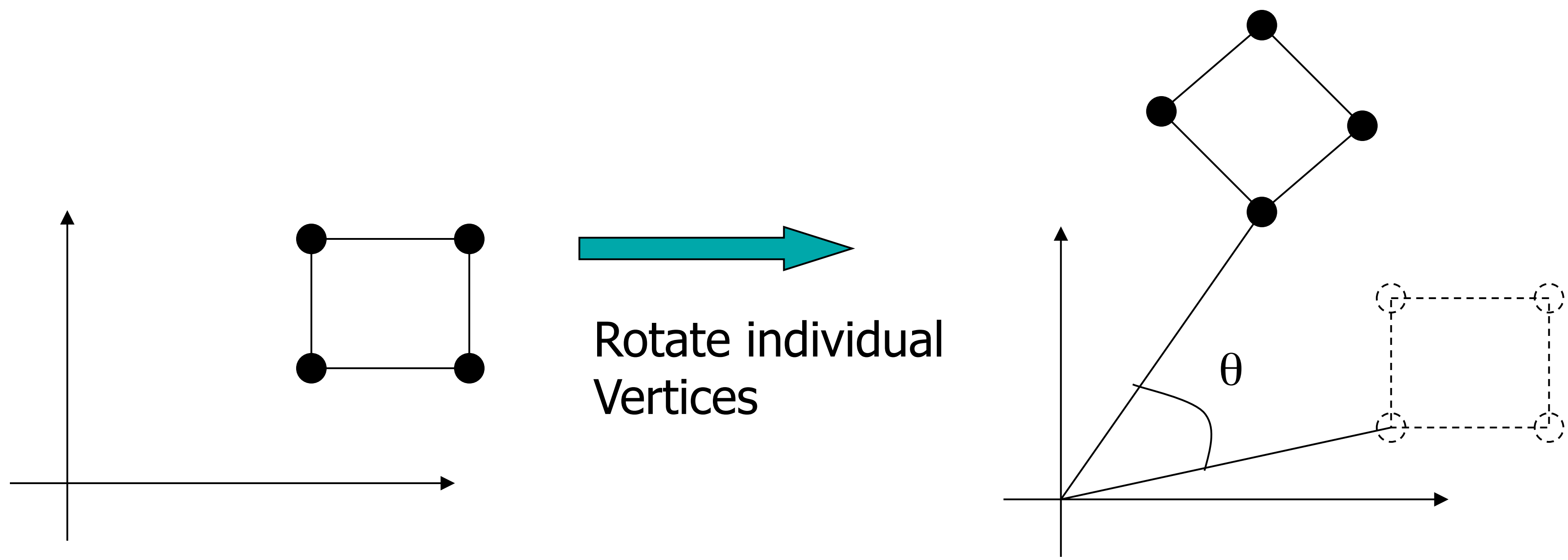
$$\Downarrow$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



2D Rotation

- How to rotate an object with multiple vertices?



Arbitrary Rotation Center

- To rotate about arbitrary point P = (Px, Py) by θ:
  - Translate object by T(-Px, -Py) so that P coincides with origin
  - Rotate the object by R(θ)
  - Translate object back: T(Px, Py)
- In matrix form
  - T(Px,Py) R(θ) T(-Px,-Py) * P

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & Px \\ 0 & 1 & Py \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -Px \\ 0 & 1 & -Py \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Similar for arbitrary scaling anchor

Composing Transformations

- Composing transformations
  - Applying several transforms in succession to form one overall transformation
- Example
  - **M1 X M2 X M3 X P**

  where M1, M2, M3 are transform matrices applied to P
- Be careful with the order!
- For example
  - Translate by (5, 0), then rotate 60 degrees is NOT same as
  - Rotate by 60 degrees, then translate by (5, 0)

OpenGL Transformations

- Designed for 3D
- For 2D, simply ignore *z* dimension
- Translation:
  - `glTranslated( tx, ty, tz )`
  - `glTranslated( tx, ty, 0 )` => for 2D
- Rotation:
  - `glRotated( angle, Vx, Vy, Vz )`
  - `glRotated( angle, 0, 0, 1 )` => for 2D
- Scaling:
  - `glScaled( sx, sy, sz )`
  - `glScaled( sx, sy, 0 )` => for 2D

3D Transformations

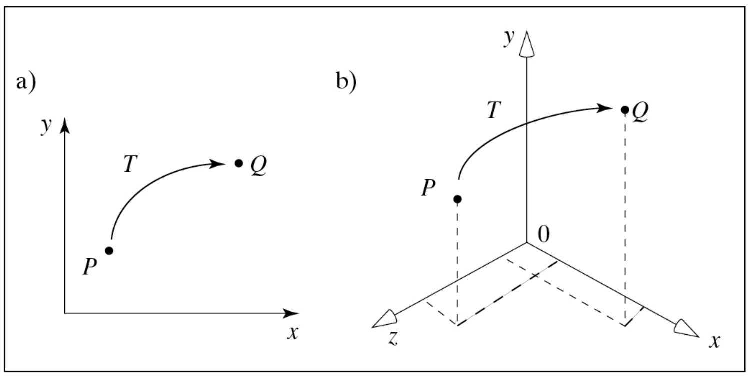
- Affine transformations
  - Mappings of points to new points that retain certain relationships
  - Lines remain lines
  - Several transformations can be combined into a single matrix
- Two ways to think about transformations
  - Object transformations
    - All points of an object are transformed
  - Coordinate transformations
    - The coordinate system is transformed, and models remain defined relative to this

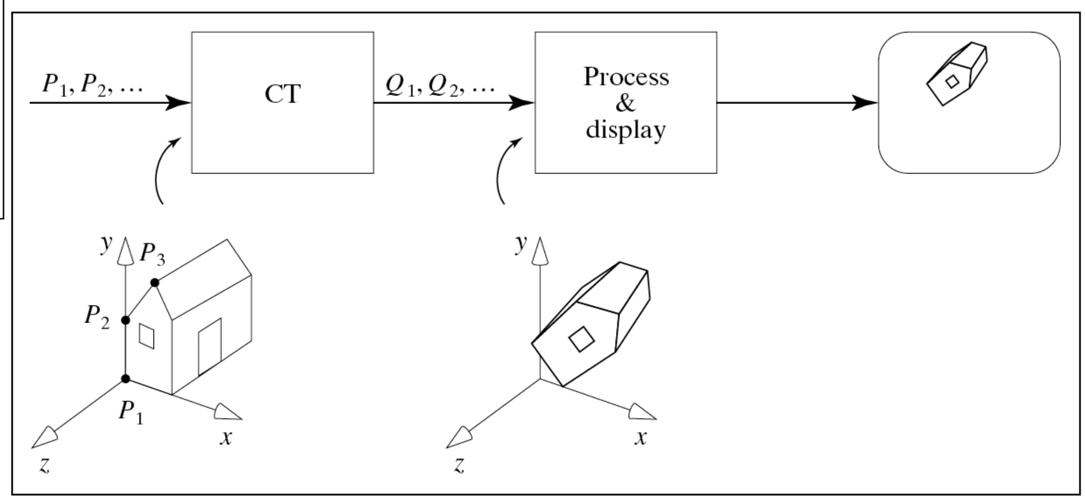
3D Transformations (cont.)

- Scale
  - **`glScaled( sx, sy, sz )`**: Scale object by (sx, sy, sz)
- Translate
  - **`glTranslated( dx, dy, dz )`**: Translate object by (dx, dy, dz)
- Rotate
  - **`glRotated( angle, ux, uy, uz )`**: Rotate by angle about an axis passing through origin and (ux, uy, uz)
- OpenGL
  - Creates a matrix for each transformation
  - Multiplies matrices together to form a single, combined matrix
  - Transformation matrix is called ***modelview matrix***

Example: Translation

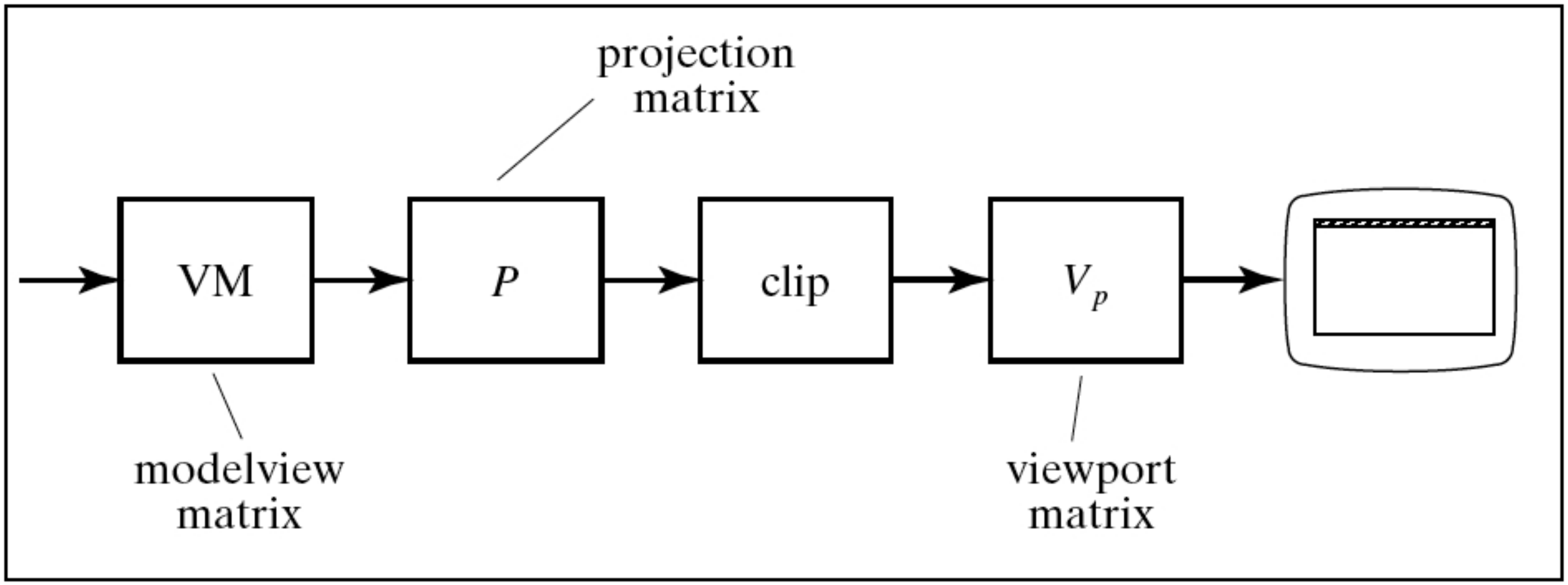
- The vertices of an object are mapped to new points
  - Similarly for scaling and rotation





OpenGL Matrices

- Graphics pipeline takes all vertices through a series of operations



OpenGL Matrices and the Pipeline

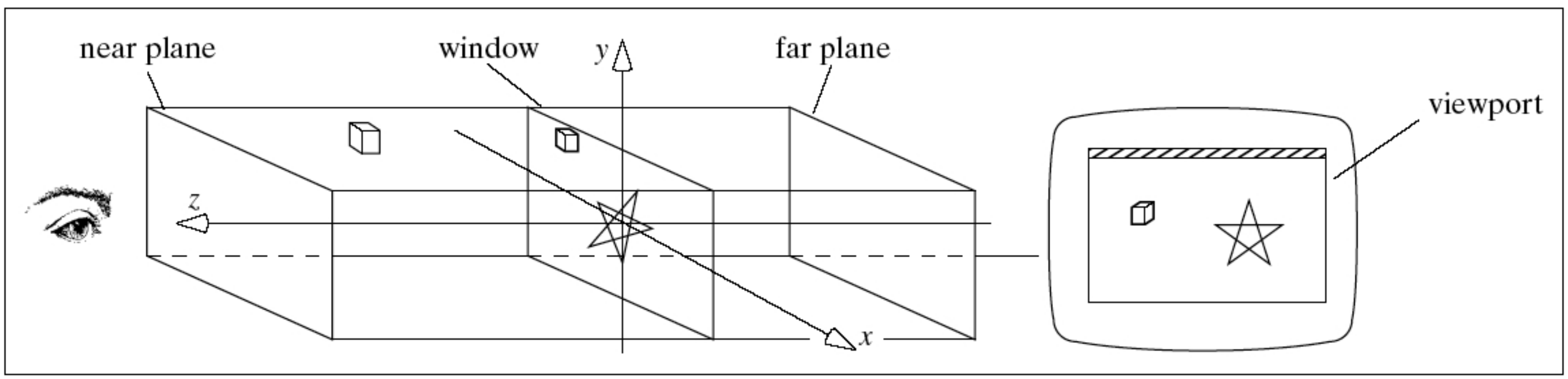
- OpenGL uses three matrices for geometry
  - Modelview matrix
  - Projection matrix
  - Viewport matrix
- Modelview matrix
  - Combination of modeling matrix *M* and camera transforms *V*
- Other OpenGL matrices include texture and color matrices
- **`glMatrixMode`** command selects matrix mode
- **`glMatrixMode`** parameters
  - **`GL_MODELVIEW, GL_PROJECTION, GL_TEXTURE`**, *etc.*
- May initialize matrices with **`glLoadIdentity( )`**

OpenGL Matrices and the Pipeline

- OpenGL matrix operations are 4x4 matrices
- Graphics card
  - Fast 4x4 multiplier -> tremendous speedup

View Frustum

- Side walls determined by window borders
- Other walls are programmer defined
  - Near clipping plane
  - Far clipping plane
- Transform 3D models to 2D
  - Project points/vertices inside view volume onto view window using parallel lines along z-axis



Types of Projections

- Different types of projections?
  - Different view volume shapes
  - Different visual effects
- Example projections
  - Parallel (a.k.a. *orthographic*)
  - Perspective
- Parallel is simple
- Will use this for intro, expand later

OpenGL Matrices and the Pipeline

- Projection matrix
  - Scales and shifts each vertex in a particular way
  - View volume lies inside cube of –1 to 1
  - Reverses sense of z
    - increasing z = increasing depth
  - Effectively squishes view volume down to cube centered at 1
- Clipping in 3D then eliminates portions outside view frustum
- Viewport matrix:
  - Maps surviving portion of block (cube) into a 3D viewport
  - Retains a measure of the depth of a point