

# On edge colorings with at least $q$ colors in every subset of $p$ vertices

Gábor N. Sárközy, Stanley Selkow

Computer Science Department  
Worcester Polytechnic Institute  
Worcester, MA 01609  
gsarkozy@cs.wpi.edu, sms@cs.wpi.edu

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## Abstract

For fixed integers  $p$  and  $q$ , an edge coloring of  $K_n$  is called a  $(p, q)$ -coloring if the edges of  $K_n$  in every subset of  $p$  vertices are colored with at least  $q$  distinct colors. Let  $f(n, p, q)$  be the smallest number of colors needed for a  $(p, q)$ -coloring of  $K_n$ . In [3] Erdős and Gyárfás studied this function if  $p$  and  $q$  are fixed and  $n$  tends to infinity. They determined for every  $p$  the smallest  $q$  ( $= \binom{p}{2} - p + 3$ ) for which  $f(n, p, q)$  is linear in  $n$  and the smallest  $q$  for which  $f(n, p, q)$  is quadratic in  $n$ . They raised the question whether perhaps this is the only  $q$  value which results in a linear  $f(n, p, q)$ . In this paper we study the behavior of  $f(n, p, q)$  between the linear and quadratic orders of magnitude. In particular we show that that we can have at most  $\log p$  values of  $q$  which give a linear  $f(n, p, q)$ .

## 1 Introduction

### 1.1 Notations and definitions

For basic graph concepts see the monograph of Bollobás [1].  $V(G)$  and  $E(G)$  denote the vertex-set and the edge-set of the graph  $G$ .  $K_n$  is the complete graph on  $n$  vertices. In this paper  $\log n$  denotes the base 2 logarithm.  $pr(n)$  denotes the parity of the natural number  $n$ , so it is 1 if  $n$  is odd and 0 otherwise.

### 1.2 Edge colorings with at least $q$ colors in every subset of $p$ vertices

The following interesting concepts were created by Erdős, Elekes and Füredi (see [2]) and then later studied by Erdős and Gyárfás in [3] (see also [4]). For fixed integers  $p$  and  $q$  an edge coloring of  $K_n$  is called a  $(p, q)$ -coloring if in every subset of  $p$  vertices at least  $q$

distinct colors appear on the edges. Let  $f(n, p, q)$  be the smallest number of colors needed for a  $(p, q)$ -coloring of  $K_n$ . It will be always assumed that  $p \geq 3$  and  $2 \leq q \leq \binom{p}{2}$ . We restrict our attention to the case when  $p$  and  $q$  are fixed and  $n$  tends to infinity. The study of  $f(n, p, q)$  leads to many interesting and difficult problems. For example determining  $f(n, p, 2)$  is equivalent to determining classical Ramsey numbers for multicolorings.

Among many other interesting results and problems in [3] Erdős and Gyárfás determined for every  $p$  the smallest  $q$  ( $q_{lin} = \binom{p}{2} - p + 3$ ) for which  $f(n, p, q)$  is linear in  $n$  and the smallest  $q$  ( $q_{quad} = \binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 2$ ) for which  $f(n, p, q)$  is quadratic in  $n$ . They raised the striking question if  $q_{lin}$  is the only  $q$  value which results in a linear  $f(n, p, q)$ . In this paper we study the behavior of  $f(n, p, q)$  between the linear and quadratic orders of magnitude, so for  $q_{lin} \leq q \leq q_{quad}$ . In particular we show that that we can have at most  $\log p$  values of  $q$  which give a linear  $f(n, p, q)$ .

In order to state our results, first we need some definitions. We define the following two strictly decreasing sequences  $a_i$  and  $b_j$  of positive integers with  $a_0 = p$ . Roughly speaking  $a_{i+1} = \lfloor \frac{a_i}{2} \rfloor$  but for every second odd  $a_i$  we have to add 1.

The two sequences are defined recursively. Assuming  $a_0, a_1, \dots, a_i$  are already defined, the sequence  $b_1, b_2, \dots, b_{i'}$  is just the subsequence consisting of the odd  $a_i$ -s which are greater than 1. Then we define

$$a_{i+1} = \begin{cases} \lfloor \frac{a_i}{2} \rfloor & \text{if } a_i = b_j \text{ for an even } j \\ \lfloor \frac{a_i}{2} \rfloor + 1 & \text{otherwise} \end{cases}$$

Furthermore if  $a_{i+1}$  is odd and greater than 1, then  $b_{i'+1} = a_{i+1}$ .

So, for example, if  $p = 2^k$ , the sequence of  $a_i$ -s is just all the powers of 2 from  $p$  to 1, while there are no  $b_j$ -s. Let  $l_p$  be the smallest integer for which  $a_{l_p} = 1$ .

We will need the following simple lemma.

**Lemma 1.** *For  $1 \leq i \leq l_p$ , we have*

$$a_i \leq \frac{p}{2^i} + 1 - \frac{1}{2^{i-1}} < \frac{p}{2^i} + 1. \tag{1}$$

The simple inductive proof is given in the next section. This lemma immediately gives the bound

$$l_p \leq \lceil \log p \rceil. \tag{2}$$

Our main result is the following.

**Theorem 1.** *For positive integers  $p$ ,  $1 \leq k \leq l_p$ , if  $q \geq q_{lin} + a_k + k - 1$ , then*

$$f(n, p, q) > \frac{1}{4p^2} n^{\frac{2^k}{2^k-1}}.$$

Using Lemma 1, we immediately get the following.

**Corollary 2.** For positive integers  $p$ ,  $1 \leq k \leq l_p$ , if  $q \geq q_{lin} + \frac{p}{2^k} + k$ , then

$$f(n, p, q) > \frac{1}{4p^2} n^{\frac{2^k}{2^k-1}}.$$

Note that this is not far from the truth. (In fact, for  $k = 1$  it gives the right order of magnitude, namely quadratic.) Indeed, from the general probabilistic upper bound of [3], we get the following.

**Theorem 3.** ([3]) For positive integers  $p$ ,  $1 \leq k \leq l_p$ , if  $q \leq q_{lin} + \frac{p}{2^k} - \frac{1}{2^{k-1}}$ , then

$$f(n, p, q) \leq c_{p,q} n^{\frac{2^k}{2^k-1}},$$

where  $c_{p,q}$  depends only on  $p$  and  $q$ .

Another corollary of the lower bound in Theorem 1 ( $k = l_p$  and we use (2)) is that we can have at most  $\log p$  values with a linear  $f(n, p, q)$ .

**Corollary 4.** If  $q \geq q_{lin} + \log p$ , then

$$f(n, p, q) > \frac{1}{4p^2} n^{\frac{2^{l_p}}{2^{l_p}-1}}.$$

We have roughly a “gap” of size at most  $k$  in the values of  $q$  between the lower bound of Corollary 2 and the upper bound of Theorem 3. It would be desirable to close this gap. We believe, as is often the case, that the probabilistic upper bound (Theorem 3) is closer to the truth. First we give some preliminary facts in the next section. Then in Section 3 we prove Theorem 1.

## 2 Preliminaries

**Proof of Lemma 1:** For the two sequences  $a_i$  and  $b_j$  defined earlier, we have

$$2a_{i+1} = a_i + \begin{cases} 0 & \text{if } a_i \neq b_j \text{ for any } j \text{ (if } a_i \text{ is even)} \\ 1 & \text{if } a_i = b_j \text{ for an even } j \\ (-1) & \text{if } a_i = b_j \text{ for an odd } j \end{cases} \quad (3)$$

$l_p$  is the smallest integer for which  $a_{l_p} = 1$ . To prove Lemma 1 we use induction on  $i = 1, 2, \dots, l_p$ . It is true for  $i = 1$ . Assume that it is true for  $i$  and then for  $i + 1$  from the definition of  $a_{i+1}$  we get

$$a_{i+1} \leq \frac{a_i + 1}{2} \leq \frac{\frac{p}{2^i} + 1 - \frac{1}{2^{i-1}} + 1}{2} = \frac{p}{2^{i+1}} + 1 - \frac{1}{2^i},$$

and thus proving Lemma 1.  $\square$

Let  $l'_p$  be the number of  $b_j$ -s among  $a_0, a_1, \dots, a_{l_p-1}$ . We introduce the following indicator for  $0 \leq i \leq l_p - 1$ .

$$\delta_i = \begin{cases} 1 & \text{if } b_{j-1} > a_i \geq b_j \text{ for an odd } j > 1, \text{ or if } a_i \geq b_1, \text{ or for even } l'_p \text{ if } a_i < b_{l'_p} \\ 0 & \text{otherwise} \end{cases}$$

We will need the following.

**Lemma 2.** For any  $0 \leq i \leq l_p - 1$

$$\sum_{j=0}^i a_{l_p-j} = a_{l_p-i-1} - \delta_{l_p-i-1} - pr(l'_p). \quad (4)$$

**Proof:** We use induction on  $i = 0, 1, \dots, l_p - 1$ . (4) is true for  $i = 0$ , since  $a_{l_p} = 1$  and  $a_{l_p-1} = 1 + \delta_{l_p-1} + pr(l'_p)$ .

Assuming that (4) is true for  $i$ , for  $i + 1$  using (3) we get

$$\sum_{j=0}^{i+1} a_{l_p-j} = \sum_{j=0}^i a_{l_p-j} + a_{l_p-i-1} = 2a_{l_p-i-1} - \delta_{l_p-i-1} - pr(l'_p) = a_{l_p-i} - \delta_{l_p-i} - pr(l'_p),$$

proving the lemma.  $\square$

From this we get:

**Lemma 3.** For any  $1 \leq k \leq l_p$

$$\sum_{j=1}^k a_j \geq a_0 - a_k - 1 = p - a_k - 1.$$

**Proof:**

$$\sum_{j=1}^k a_j = \sum_{j=0}^{l_p-1} a_{l_p-j} - \sum_{j=0}^{l_p-k-1} a_{l_p-j} = a_0 - \delta_0 - a_k + \delta_k \geq a_0 - a_k - 1.$$

$\square$

### 3 Proof of Theorem 1

Let  $1 \leq k \leq l_p$  and  $q \geq q_{lin} + a_k + k - 1$ . Denote

$$h = h(n, k) = \frac{1}{4p^2} n^{\frac{2k}{2^k-1}}. \quad (5)$$

Assume indirectly that there is a  $(p, q)$ -coloring of  $K_n$  with at most  $h$  colors. From this assumption we will get a contradiction.

Consider a fixed  $(p, q)$ -coloring of  $K_n$  with at most  $h$  colors. We will find a sequence of monochromatic matchings  $M_1, M_2, \dots, M_k$  in  $K_n$ . To obtain  $M_1$  observe that there

is a color class (denoted by  $C_1$ ) in  $K_n$  which contains at least  $\frac{\binom{n}{2}}{h}$  edges. In  $C_1$  all the connected components have size at most  $p-1$ , since otherwise we immediately have a  $K_p$  with fewer than  $q$  colors, a contradiction. Then in  $C_1$  we can clearly choose a matching  $M_1$  (for example by taking one edge from each non-trivial component) which is even and has size at least

$$\frac{\binom{n}{2}}{ph}.$$

Partition the vertices spanned by  $M_1$  into  $A_1$  and  $B_1$ , so  $M_1$  is a matching between  $A_1$  and  $B_1$ . Halve the vertices of  $A$  arbitrarily and denote one of the halves by  $A'_1$ . Denote by  $B'_1$  the set of vertices in  $B_1$  which are not matched to vertices in  $A'_1$  by  $M_1$ . Consider the complete bipartite graph between  $A'_1$  and  $B'_1$  and the color class (denoted by  $C_2$ , not necessarily distinct from  $C_1$ ) which contains the most edges in it.

Again from these edges in  $C_2$  we can choose a matching  $M_2$  with partite sets  $A_2, B_2$  of even size at least

$$\frac{\left(\frac{|M_1|}{2}\right)^2}{ph}.$$

We continue in this fashion. Assume that  $M_i = (A_i, B_i)$  is already defined. Denote an arbitrary half of the endvertices of  $M_i$  in  $A_i$  by  $A'_i$ . The set of endvertices of the edges of  $M_i$  in  $B_i$  which are not matched to vertices in  $A'_i$  is denoted by  $B'_i$ . Consider the complete bipartite graph between  $A'_i$  and  $B'_i$  and the color class (denoted by  $C_{i+1}$ ) which contains the most edges in it.

From these edges in  $C_{i+1}$  we can choose a matching  $M_{i+1}$  of even size at least

$$\frac{\left(\frac{|M_i|}{2}\right)^2}{ph}.$$

Then by induction we have

$$|M_i| \geq \frac{n^{2^i}}{(4ph)^{2^i-1}}.$$

Indeed, this is true for  $i = 1$

$$|M_1| > \frac{n^2}{4ph}.$$

For  $i + 1$  we get

$$|M_{i+1}| \geq \frac{\left(\frac{|M_i|}{2}\right)^2}{ph} \geq \frac{\left(\frac{n^{2^i}}{2(4ph)^{2^i-1}}\right)^2}{ph} = \frac{n^{2^{i+1}}}{(4ph)^{2^{i+1}-1}}.$$

This and (5) implies that  $|M_i| \geq p \geq a_i$ ,  $1 \leq i \leq k$  and thus the matchings  $M_1, M_2, \dots, M_k$  can be chosen.

Next using these matchings  $M_i$  we choose a  $K_p$  such that it contains at most  $q-1$  colors, a contradiction. For this purpose we will find another sequence of matchings  $M'_i$  such that  $M'_i \subset M_i$ ,  $|M'_i| = a_i$  for  $1 \leq i \leq k$  and  $\left|\bigcup_{i=1}^k V(M'_i)\right| \leq p$ .

$M'_k$  is just a set of  $a_k$  arbitrary edges from  $M_k$ . Assume that  $M'_k, \dots, M'_{i+1}$  are already defined and now we define  $M'_i$ . We consider the  $2a_{i+1}$  vertices in  $V(M'_{i+1})$  and the edges of  $M_i$  incident to these vertices. We have four cases.

**Case 1:** If  $2a_{i+1} = a_i$  (so we have the first case in (3)), then this is  $M'_i$ .

**Case 2:** If  $2a_{i+1} = a_i + 1$  (second case in (3)), so  $a_i = b_j$  for an even  $j$ , then we remove one of the edges from this set incident to a vertex in  $V(M'_{i+1}) \cap A$  to get  $M'_i$ . Furthermore, we mark this vertex in  $V(M'_{i+1}) \cap A$  which is not covered by  $M'_i$ . This marked vertex is going to be covered only by  $M'_{i'}$  if  $a_{i'} = b_{j-1}$  (unless  $i' = 0$ ).

**Case 3:** If  $2a_{i+1} = a_i - 1$  (third case in (3)) and there is no marked vertex at the moment, then to get  $M'_i$  we add one arbitrary edge of  $M_i$  to these  $2a_{i+1}$  edges.

**Case 4:** Finally, if  $2a_{i+1} = a_i - 1$  and there is a marked vertex then to get  $M'_i$  we add to these  $2a_{i+1}$  edges the edge of  $M_i$  incident to the marked vertex and we “unmark” this vertex.

We continue in this fashion until  $M'_k, \dots, M'_1$  are defined. Then  $|\cup_{i=1}^k V(M'_i)| = p$  or  $p - 1$ . Note that it can be  $p - 1$  only if  $a_0 = p = b_1$  is odd, and there is no other odd  $a_i$  among  $a_1, a_2, \dots, a_{k-1}$ . In this case we add one more arbitrary vertex to get the  $K_p$ , otherwise  $\cup_{i=1}^k V(M'_i)$  is the  $K_p$ .

By the above construction this  $K_p$  contains  $a_i$  edges from the matching  $M_i$  (and thus from color class  $C_i$ ) for  $1 \leq i \leq k$ .

Now since Lemma 3 implies

$$\sum_{j=1}^k (a_j - 1) \geq p - a_k - 1 - k,$$

thus the number of colors used in this  $K_p$  is at most

$$\binom{p}{2} - p + a_k + k + 1 \leq q - 1,$$

a contradiction. This completes the proof of Theorem 1.  $\square$

## References

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