Tripartite Ramsey Numbers for Paths

András Gyárfás,[1] Miklós Ruszinkó,[1] Gábor N. Sárközy,[1,2] and Endre Szemerédi[3]

[1] COMPUTER AND AUTOMATION RESEARCH INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, P.O. BOX 63
BUDAPEST, HUNGARY, H-1518
E-mail: gyarfas@sztaki.hu
ruszinko@sztaki.hu

[2] COMPUTER SCIENCE DEPARTMENT
WORCESTER POLYTECHNIC INSTITUTE
WORCESTER, MASSACHUSETTS, 01609
E-mail: gsarkozy@cs.wpi.edu;
gsarkozy@sztaki.hu

[3] COMPUTER SCIENCE DEPARTMENT
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY, 08903
E-mail: szemered@cs.rutgers.edu

Received February 19, 2005; Revised October 20, 2006

Published online 14 February 2007 in Wiley InterScience(www.interscience.wiley.com).
DOI 10.1002/jgt.20231

Abstract: In this article, we study the tripartite Ramsey numbers of paths. We show that in any two-coloring of the edges of the complete tripartite graph $K(n, n, n)$ there is a monochromatic path of length $(1 - o(1))2n$. Since $R(P_{2n+1}, P_{2n+1}) = 3n$, this means that the length of the longest monochromatic path is about the same when two-colorings of K_{3n} and $K(n, n, n)$ are considered. © 2007 Wiley Periodicals, Inc. J Graph Theory 55: 164–174, 2007

Contract grant sponsor: OTKA (to M.R.); Contract grant numbers: T038198 and T046234; Contract grant sponsor: National Science Foundation (to G.N.S.); Contract grant number: DMS-0456401.

Keywords: *regularity lemma; paths; Ramsey numbers*

1. INTRODUCTION

A. Ramsey Numbers for Paths

If G_1 and G_2 are graphs, then the Ramsey number $R(G_1, G_2)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into 2 disjoint color classes giving graphs H_1 and H_2, then one of the subgraphs H_i ($i = 1, 2$) has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's original paper [13]. The number $R(G_1, G_2)$ is called the Ramsey number for the graphs G_1 and G_2. The determination of these numbers has turned out to be remarkably difficult in certain cases (see e.g., [4] or [12] for results and problems). In this article, we consider the case when each G_i is a path P_n on n vertices. A theorem of Gerencsér and Gyárfás [3] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor. \tag{1}$$

Our main result vaguely says that asymptotically this result (perhaps surprisingly) does not change if, instead of a complete graph, a complete graph with "three large holes" (i.e., a balanced tripartite graph) is colored. More precisely:

Theorem 1. *In any two-coloring of the edges of the complete tripartite graph $K(n, n, n)$ there is a monochromatic $P_{(1-o(1))2n}$.*

We note that recently there has been some interest in tripartite versions of classical results, see e.g., the result of Magyar and Martin [11], a tripartite version of the Corrádi-Hajnal Theorem.

In the proof of Theorem 1 the notion of a *connected matching* plays a central role; this is a matching M in a graph G such that all edges of M are in the same connected component of G. The approach was suggested by Łuczak [10] and applied in [2,5].

Sections 2 and 3 provide our main tools including the Regularity Lemma. Then in Section 4, we prove our main lemma (Lemma 7) which states that in any two-coloring of a $(1-\varepsilon)$-dense tripartite graph $G(l, l, l)$ there is a monochromatic connected matching covering almost $2l$ vertices. Finally in Section 5, we show how Lemma 7 implies Theorem 1.

It is worth noting that Theorem 1 remains true (with the proof of this article) if $o(n^2)$ edges are missing from $K(n, n, n)$. It seems reasonable to conjecture that the following (sharp) version also holds: if the edges of $K(n, n, n)$ are two-colored, then there exists a monochromatic P_{2n+1}—this would generalize (1) for odd n.

B. Notation and Definitions

For basic graph concepts see the monograph of Bollobás [1]. Disjoint union of sets will be sometimes denoted by +. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. Usually, G_n is a graph with n vertices, $G(k, k, k)$ is a tripartite graph with k-element vertex classes. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. K_n is the complete graph on n vertices, $K(n_1, \ldots, n_k)$ is the complete k-partite graph with classes containing $n_1, \ldots, n_k$ vertices, P_n (C_n) is the path (cycle) with n vertices. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $\Gamma(v)$ is the set of neighbors of $v \in V$. Hence, the size of $\Gamma(v)$ is $|\Gamma(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. For a vertex $v \in V$ and set $U \subset V - \{v\}$, we write $deg(v, U)$ for the number of edges from v to U. A graph G_n is *γ-dense* if it has at least $\gamma\binom{n}{2}$ edges. $G(k, k, k)$ is γ-dense if it contains at least $3\gamma k^2$ edges. When A, B are disjoint subsets of $V(G)$, we denote by $e_G(A, B)$ the number of edges of G with one endpoint in A and the other in B. For non-empty A and B,

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}$$

is the *density* of the graph between A and B.

Definition 1. *The bipartite graph $G = (A, B, E)$ is (ε, G)-regular if*

$$X \subset A,\ Y \subset B,\ |X| > \varepsilon|A|,\ |Y| > \varepsilon|B| \quad \textit{imply} \quad |d_G(X, Y) - d_G(A, B)| < \varepsilon,$$

otherwise it is ε-irregular.

2. THE REGULARITY LEMMA

In the proof, a two-color version of the Regularity Lemma plays a central role.

Lemma 1 (Regularity Lemma [14])**.** *For every positive ε and positive integer m, there are positive integers M and n_0 such that for $n \geq n_0$ the following holds. For all graphs G_1 and G_2 with $V(G_1) = V(G_2) = V$, $|V| = n$, there is a partition of V into $l + 1$ classes (clusters)*

$$V = V_0 + V_1 + V_2 + \cdots + V_l$$

such that

- $m \leq l \leq M$
- $|V_1| = |V_2| = \cdots = |V_l|$
- $|V_0| < \varepsilon n$
- *apart from at most $\varepsilon\binom{l}{2}$ exceptional pairs, the pairs $\{V_i, V_j\}$ are (ε, G_s)-regular for $s = 1, 2$.*

For an extensive survey on different variants of the Regularity Lemma see [8]. Note also that if we apply the Regularity Lemma for a balanced tripartite graph G, we can guarantee that for each cluster that is not V_0, all vertices of the cluster belong to the same partite class of G (see e.g., [11]).

We will also use the following simple property of (ε, G)-regular pairs.

Lemma 2. *Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = m \geq 45$. Furthermore, let $e_G(V_1, V_2) \geq m^2/4$ and the pair $\{V_1, V_2\}$ be (ε, G)-regular for $0 < \epsilon < 0.01$. Then for every pair of vertices $v', v'' \in V(G)$, where $deg(v'), deg(v'') \geq m/5$, G contains a path of length at least $(1 - 5\varepsilon)2m$ connecting v' and v''.*

This lemma is used by Łuczak in [10]. Lemma 2 (with somewhat weaker parameters) also follows from the much stronger Blow-up Lemma (see [6] and [7]).

3. FURTHER GRAPH THEORY TOOLS

A set M of pairwise disjoint edges of a graph G is called a matching. The size $|M|$ of a maximum matching is the matching number, $\nu(G)$. A key notion in our approach is the notion of a connected matching. A matching M is *connected* in G if all edges of M are in the same component of G. The following result is often referred to as the Tutte–Berge formula (see e.g., in [9] Theorem 3.1.14). We shall use $c(G)$ and $c_o(G)$ for the number of components and odd components of a graph G and def(G), the deficiency of G, is defined as $|V(G)| - 2\nu(G)$.

Lemma 3. *For any graph G, $def(G) = \max\{c_o(G \setminus S) - |S|\}$ where the maximum is taken over all $S \subseteq V(G)$.*

We also need the following obvious property of maximum matchings.

Lemma 4. *Suppose $M = \{e_1, \ldots, e_k\}$ is a maximum matching in a graph G. Then $V(G) \setminus V(M)$ spans an independent set and one can select one end point x_i of each e_i so that for each i, $1 \leq i \leq k$, there is at most one edge in G from x_i to $V(G) \setminus V(M)$.*

For a tripartite graph $G = G(l, l, l)$, we shall work with its *tripartite complement*, $\overline{G}$, defined as the graph we obtain from the usual complement after deleting all edges within the partite classes. The next lemmas collect some simple properties of graphs of high density.

Lemma 5. *Assume that $G = G(l, l, l)$ is a $(1 - \varepsilon)$-dense tripartite graph. Then G has a tripartite subgraph $H = H(k, k, k)$ with $k \geq (1 - 2\sqrt{\varepsilon})l$ such that: A. $\Delta(\overline{H}) < 2\sqrt{\varepsilon}l$; B. $\delta(H) \geq (2 - 6\sqrt{\varepsilon})l$; C. H is $(1 - 3\sqrt{\varepsilon})$-dense.*

Proof. If G has p vertices in the same partite class with degree at least $2\sqrt{\varepsilon}l$ in $\overline{G}$, then $\overline{G}$ has at least $p2\sqrt{\varepsilon}l$ edges. Therefore $p2\sqrt{\varepsilon}l \leq 3\varepsilon l^2$, implying $p \leq \frac{3}{2}\sqrt{\varepsilon}l < 2\sqrt{\varepsilon}l$. Removing these p vertices from each partite class, the remaining vertices induce the subgraph H. Properties A. and B. are obvious and C. follows from

$$|E(H)| \geq \frac{|V(H)|\delta(H)}{2} \geq \frac{|V(H)|(2-6\sqrt{\varepsilon})l}{2} = 3k(1-3\sqrt{\varepsilon})l \geq 3(1-3\sqrt{\varepsilon})k^2.$$

■

Lemma 6. *Assume $\Delta(\overline{G_n}) < \sqrt{\varepsilon}n$ and $H = [A, B]$ is a bipartite subgraph of G_n with $2\sqrt{\varepsilon}n < |A| \leq |B|$. Then H is a connected subgraph of G_n and contains a matching of size at least $|A| - \sqrt{\varepsilon}n$. Moreover, if only $2\sqrt{\varepsilon}n < |B|$ and $A \neq \emptyset$ is assumed then there is a subgraph H' which is connected and covers A and all but at most $\sqrt{\varepsilon}n$ vertices of B.*

Proof. Two vertices in A (B) have a common neighbor in B (A). Also if $a \in A$, $b \in B$ then any neighbor of a and b have a common neighbor in A. Thus H is a connected subgraph. Moreover any maximum matching M misses fewer than $\sqrt{\varepsilon}n$ vertices of A. The statement about H' follows by fixing a vertex $a \in A$ and H' is obtained by deleting from B the vertices nonadjacent to A. ■

4. LARGE MONOCHROMATIC CONNECTED MATCHINGS IN BALANCED TRIPARTITE GRAPHS

A monochromatic (say red) matching in a colored complete or almost complete graph is called *connected* if its edges are all in the same monochromatic connected red component. For example, if K_4 is three-colored so that each color class has two disjoint edges (factorization of K_4) then the largest monochromatic matching has two edges, but the largest connected monochromatic matching has only one edge.

In our main lemma, we show that we can find large monochromatic connected matchings in balanced tripartite graphs.

Lemma 7. *Suppose that $\sqrt{\varepsilon} < \frac{1}{132}$ and $l \geq \frac{1}{\sqrt{\varepsilon}(1-2\sqrt{\varepsilon})}$. Then every two-coloring of a $(1-\varepsilon)$-dense tripartite graph $G(l, l, l)$ contains a monochromatic connected matching covering at least $(2 - 532\sqrt{\varepsilon})l$ vertices.*

Proof. By Lemma 5 select $H = H(k, k, k) \subseteq G(l, l, l)$ with $\Delta(\overline{H}) < 2\sqrt{\varepsilon}l$, $\delta(H) \geq (2 - 6\sqrt{\varepsilon})l$ and $k \geq (1 - 2\sqrt{\varepsilon})l$. Let V_i denote the partite classes of H.

We claim first that there is a set $Z \subseteq V(H)$ such that $|Z| \geq 3k(1 - 6\sqrt{\varepsilon})$ and in one of the two colors, the edges of this color inside Z determine only one nontrivial component. This color is called the color of Z.

To prove the claim, select a largest monochromatic, say, red component C_1. Set $R_i = V_i \cap V(C_1)$, $S_i = V_i \setminus V(C_1)$. If $V(C_1)$ covers two of the V_i-s then $Z = V(H)$ satisfies the claim with the red color. Thus at least two of the S_i-s are nonempty. Furthermore, by the choice of C_1, $|C_1| \geq (1 - 3\sqrt{\varepsilon})l$ by B. of Lemma 5.

Call a set small if it has less than $4\sqrt{\varepsilon}l$ elements, otherwise it is large. The condition $\sqrt{\varepsilon} < \frac{1}{10}$ ensures that $8\sqrt{\varepsilon}l \leq (1 - 2\sqrt{\varepsilon})l \leq k$, thus at least one of $|R_i|, |S_i|$ is large. Lemma 6 with $3l$ in the role of n and with $\frac{2}{3}\sqrt{\varepsilon}$ in the role of $\sqrt{\varepsilon}$ can be applied to H since $\Delta(\overline{H}) < 2\sqrt{\varepsilon}l = \frac{2}{3}\sqrt{\varepsilon}3l$. This gives that $[R_i, S_j]$

is connected in blue if both R_i, S_j are large. Else it has a blue component covering all but at most $2\sqrt{\varepsilon}l$ vertices of the larger part if only one of them is large and the other is nonempty. With these remarks in mind we have the following cases.

If all the three S_i-s are small then C_1 misses only these small sets, thus

$$|C_1| \geq 3k - 12\sqrt{\varepsilon}l \geq 3(1-2\sqrt{\varepsilon})l - 12\sqrt{\varepsilon}l = (1-6\sqrt{\varepsilon})3l \geq (1-6\sqrt{\varepsilon})3k \quad (2)$$

and the claim follows with $Z = V(C_1)$ in color red.

If exactly one S_i, say, S_1 is large then R_2, R_3 are both large. Lemma 6 implies that $C_2 = S_1 \cup R_2 \cup R_3$ is connected in blue. If R_1 is small then C_2 works as Z in blue with the same estimate as (2). If R_1 is large then it is joined to C_2 through S_2 or through S_3, whichever is nonempty. Thus (2) works with reserve.

If exactly two S_i-s are large, say, S_1, S_2, then S_3 is small implying that R_3 is large. Lemma 6 ensures that $C_3 = S_1 \cup S_2 \cup R_3$ is connected in blue. Then, applying Lemma 6 repeatedly, R_1, R_2 join to C_3. Thus $Z = V(H)$ works in color blue.

If all S_i-s are large we use that some R_i, say, R_1 is large,

$$|R_1| \geq \frac{1}{3}|C_1| \geq \frac{1}{3}(1-3\sqrt{\varepsilon})l \geq 4\sqrt{\varepsilon}l$$

if $\sqrt{\varepsilon} < \frac{1}{15}$. Then, $R_1 \cup S_1 \cup S_2 \cup S_3$ is connected in blue and $R_2 \cup R_3$ is absorbed into that blue component. Thus, in this case $Z = V(H)$ is connected in blue and the claim is proved.

Now, we define a new tripartite graph H' by deleting all edges of H inside $V(H) \setminus Z$ in the color of Z. Then we select a maximum monochromatic matching M of H' in the color of Z, say, red, it is automatic that M is connected. Apply Lemma 4 to select one end point of each edge of M, their set is denoted by U, the set of the other end points is denoted by T. Set $U_i = U \cap V_i$, $T_i = T \cap V_i$, let M_{ij} denote the edges of M going from V_i to V_j, $m_{ij} = |M_{ij}|$. Set $W_i = V_i \setminus (U_i \cup T_i)$, define H^* as the tripartite subgraph of H' induced by $V(H') \setminus (T_1 \cup T_2 \cup T_3)$.

Now Lemma 4 implies (with the convention that the exceptional red edge from each $u \in U_i$ to W_j is deleted) that the following bipartite subgraphs of H^* have only blue edges:

$$\begin{aligned}&[U_1, W_2], [U_1, W_3], [U_2, W_1], [U_2, W_3], [U_3, W_1], [U_3, W_2],\\ &[W_1, W_2], [W_1, W_3], [W_2, W_3]. \end{aligned} \quad (3)$$

From now on, we shall consider H^* as the tripartite graph defined by the (blue) edge sets of the bipartite graphs in (3), thus the edge sets of the bipartite graphs $[U_i, U_j]$ are ignored.

Claim 1. *Let B be any of the bipartite graphs $[U_i, W_j]$, $[W_i, W_j]$, $i \neq j$, $1 \leq i < j \leq 3$. Then $\Delta(\overline{B}) < 22\sqrt{\varepsilon}k$.*

Proof. Adding the losses from deleting the red edges of $V(H)\setminus Z$, one red edge per vertex from $u\in U_i$ to W_j we have

$$\begin{aligned}\Delta(\overline{B}) &\le 2\sqrt{\varepsilon}l+(|V(H)|-|Z|)+1\le 2\sqrt{\varepsilon}l+(3k-(1-6\sqrt{\varepsilon})3k)+1\\ &\le\left(\frac{2\sqrt{\varepsilon}}{1-2\sqrt{\varepsilon}}+18\sqrt{\varepsilon}\right)k+1<(3\sqrt{\varepsilon}+18\sqrt{\varepsilon})k+1\le 22\sqrt{\varepsilon}k \qquad (4)\end{aligned}$$

where we used that $\sqrt{\varepsilon}<\frac{1}{6}$ and

$$\frac{1}{1-2\sqrt{\varepsilon}}k\ge l\ge\frac{1}{\sqrt{\varepsilon}(1-2\sqrt{\varepsilon})}$$

proving Claim 1. ■

Next we establish inequalities to prove that (the blue graph) H^* has an almost spanning connected matching. Since each V_i is partitioned by U_i, T_i, W_i we have

$$|U_i|+|T_i|+|W_i|=k \qquad (5)$$

for $1\le i\le 3$. Also, we may assume

$$|M|=|U_1|+|U_2|+|U_3|=|T_1|+|T_2|+|T_3|=m_{12}+m_{13}+m_{23}<k \qquad (6)$$

otherwise M is a connected red matching of size $k\ge(1-2\sqrt{\varepsilon})l$, covering at least $(2-4\sqrt{\varepsilon})l$ vertices of G.

Notice that $|V(H^*)|=3k-|T_1\cup T_2\cup T_3|=3k-|M|>2k$. Since $|W_1|+|W_2|+|W_3|=3k-2|M|>k$, (6) gives

$$\sum_{i=1}^{3}U_i<\sum_{i=1}^{3}W_i. \qquad (7)$$

Using (6),

$$\begin{aligned}|U_1|&\le m_{12}+m_{13}<2k-m_{12}-m_{13}-2m_{23}\\ &=k-(m_{12}+m_{23})+k-(m_{13}+m_{23})=|W_2|+|W_3|\end{aligned}$$

and by symmetry we get

$$|U_1|<|W_2|+|W_3|,|U_2|<|W_1|+|W_3|,|U_3|<|W_1|+|W_2|. \qquad (8)$$

Since from (5) $|U_1|+|W_1|=k-|T_1|$ and $|U_2|+|W_2|+|U_3|+|W_3|=|V(H^*)|-(|U_1|+|W_1|)$, using (6) we get the following

$$\begin{aligned}&(|U_1|+|W_1|)-(|U_2|+|U_3|+W_2|+|W_3|)=2k-2|T_1|-|V(H^*)|\\ &=|T_2|+|T_3|-|T_1|-k<0,\end{aligned}$$

giving the last set of inequalities:

$$\begin{aligned}|U_1| + |W_1| &< |U_2| + |U_3| + |W_2| + |W_3|,\\ |U_2| + |W_2| &< |U_1| + |U_3| + |W_1| + |W_3|,\\ |U_3| + |W_3| &< |U_1| + |U_2| + |W_1| + |W_2|.\end{aligned} \tag{9}$$

Let S be an arbitrary subset of $V(H^*)$. Partition S into six parts, $S \cap U_i$, $S \cap W_i$ and let $S^* = S \cup M$ where M is the union of those U_i-s and W_i-s that satisfy $|U_i \setminus S| < 44\sqrt{\varepsilon}k$ or $|W_i \setminus S| < 44\sqrt{\varepsilon}k$. Then we have

$$|S^*| \leq |S| + 6 \times 44\sqrt{\varepsilon}k. \tag{10}$$

Claim 2. $c(H^* \setminus S^*) \leq |S^*| + 1$.

Proof. Call U_i (W_i) full, if $S^* \cap U_i = U_i$ ($S^* \cap W_i = W_i$). Observe that if $i \neq j$ and neither U_i nor W_j are full then $|U_i \setminus S^*| \geq 44\sqrt{\varepsilon}k$, $|W_j \setminus S^*| \geq 44\sqrt{\varepsilon}k$ and by Claim 1 at most $22\sqrt{\varepsilon}k$ edges are missing from the bipartite graph $B = [U_i \setminus S^*, W_j \setminus S^*]$. Therefore, by Lemma 6, B is connected. The same argument shows that $[W_i \setminus S^*, W_j \setminus S^*]$ is connected for $i \neq j$ whenever W_i, W_j are not full. This argument shows that there is at most one nontrivial component, all other components of $H^* \setminus S^*$ are trivial, that is, isolated vertices. It is obvious that removing vertices of S^* from components that are not full can not decrease the number of components of $H^* \setminus S^*$. Therefore, we may assume all sets U_i, W_i are either full or empty (i.e., $U_i \cap S^*$, $W_i \cap S^*$ are empty). This reduces the claim to check the following property of the weighted graph Q on six vertices, the skeleton of H^*, defined with vertices u_i, w_i, $1 \leq i \leq 3$ and edges $(u_i, w_j), (w_i, w_j)$, $1 \leq i < j \leq 3$ and vertex-weights $|U_i|, |W_i|$:
For every $S \subseteq V(Q)$ the total weight of the isolated points of $V(Q) \setminus S$ is smaller than the weight of S.

A moment of reflection gives that inequalities (7), (8), (9) state precisely this, finishing the proof of Claim 2. ■

Observe that for $X \subseteq X^*$, $c(G \setminus X) - |X| \leq c(G \setminus X^*) - |X^*| + 2(|X^*| - |X|)$. Using this observation, Claim 2, (10), we get

$$\begin{aligned}c(H^* \setminus S) - |S| &\leq c(H^* \setminus S^*) - |S^*| + 2(|S^*| - |S|) \leq 1 + 2(|S^*| - |S|)\\ &\leq 2 \times 6 \times 44\sqrt{\varepsilon}k\end{aligned}$$

for any $S \subseteq V(H^*)$. Applying Lemma 3 we conclude that

$$\begin{aligned}2k - 2\nu(H^*) &< |V(H^*)| - 2\nu(H^*) = \mathrm{def}(H^*) = \max\{c_o(H^* \setminus S) - |S|\}\\ &\leq \max\{c(H^* \setminus S) - |S|\} \leq 528\sqrt{\varepsilon}k\end{aligned}$$

thus H^* has a matching M_2 covering at least

$$2k - 528\sqrt{\varepsilon}k \geq (2 - 528\sqrt{\varepsilon})(1 - 2\sqrt{\varepsilon})l$$
$$= (2 - 532\sqrt{\varepsilon} + 1056\varepsilon)l \geq (2 - 532\sqrt{\varepsilon})l$$

vertices of G.

To see that M_2 is connected, we show that H^* is connected. Recall that $|W_1| + |W_2| + |W_3| > k$, so at least one $|W_i|$ is large, say, $|W_1| \geq \frac{k}{3} > 44\sqrt{\varepsilon}k$ if $\frac{1}{132} > \sqrt{\varepsilon}$. Also, by (6), all W_i-s are nonempty. Applying Lemma 6 to the bipartite graphs $[W_2, W_1]$, $[W_3, W_1]$ we get components covering W_2, W_3 and all but at most $22\sqrt{\varepsilon}$ vertices of W_1. These components must intersect by our assumption so $W_1 \cup W_2 \cup W_3$ is in the same component C of H^*. The same argument shows that U_2, U_3 also belongs to C. The only problematic set is U_1, which can be disconnected from $W_2 \cup W_3$. To avoid that, we may assume that $|M| < (1 - 11\sqrt{\varepsilon})k$, otherwise the red matching is of size $(1 - 11\sqrt{\varepsilon})k \geq (1 - 11\sqrt{\varepsilon})(1 - 2\sqrt{\varepsilon})l$, covering $(2 - 26\sqrt{\varepsilon} + 44\varepsilon)l > (2 - 26\sqrt{\varepsilon})l$ vertices of G. Then $|W_2| + |W_3| > 2k - 2|M| \geq 2k - 2(1 - 11\sqrt{\varepsilon})k = 22\sqrt{\varepsilon}k$ which (through Lemma 6) ensures that no vertex of U_1 can be disconnected from C.

It is easy to check that $2 - 532\sqrt{\varepsilon}$ is the smallest coefficient of l among the estimates. ■

5. PROOF OF THEOREM 1

We will assume that n is sufficiently large. Let $0 < \delta < 1$ be arbitrary and choose

$$\varepsilon = \frac{1}{3}\left(\frac{\delta}{225}\right)^2. \tag{11}$$

We need to show that each 2-edge coloring of $K(n, n, n)$ leads to a monochromatic path of length at least $(1 - \delta)2n$. Consider a 2-edge coloring (G_1, G_2) of $K(n, n, n)$. Let V_i denote the partite classes. Apply the two-color tripartite version of the Regularity Lemma (Lemma 1), with ε as in (11) and (by using the remark after the lemma) we can get a partition for $i = 1, 2, 3$ of $V_i = V_i^0 + V_i^1 + \cdots + V_i^l$, where $|V_i^j| = m$, $1 \leq j \leq l$, $1 \leq i \leq 3$ and $|V_i^0| < \varepsilon n$, $1 \leq i \leq 3$. We define the following *reduced graph* G^r: The vertices of G^r are p_i^j, $1 \leq j \leq l$, $1 \leq i \leq 3$, and we have an edge between vertices $p_{i_1}^{j_1}$ and $p_{i_2}^{j_2}$, $1 \leq j_1, j_2 \leq l$, $1 \leq i_1, i_2 \leq 3$, $i_1 \neq i_2$, if the pair $\{V_{i_1}^{j_1}, V_{i_2}^{j_2}\}$ is (ε, G_s)-regular for $s = 1, 2$. Thus we have a one-to-one correspondence $f : p_i^j \to V_i^j$ between the vertices of G^r and the non-exceptional clusters of the partition. Then G^r is a $(1 - \varepsilon)$-dense balanced tripartite graph. Define a 2-edge coloring (G_1^r, G_2^r) of G^r in the following way. The color of the edge between the clusters $V_{i_1}^{j_1}$ and $V_{i_2}^{j_2}$ is the majority color in the pair $\{V_{i_1}^{j_1}, V_{i_2}^{j_2}\}$.

Lemma 7 implies that in such a 2-coloring of G^r we can find a monochromatic connected matching $M = \{e_1, e_2, \ldots, e_{l_1}\}$ covering at least $(2 - 532\sqrt{3\varepsilon})l$ vertices of G^r. Assume that M is in G_1^r. Thus we have

$$\left|\cup_{i=1}^{l_1} \cup_{p\in e_i} f(p)\right| \geq (2 - 532\sqrt{3\varepsilon})(1-\varepsilon)n \geq (2 - 534\sqrt{3\varepsilon})n. \tag{12}$$

Furthermore, define $f(e_i) = (C_1^i, C_2^i)$ for $1 \leq i \leq l_1$ where C_1^i, C_2^i are the clusters assigned to the end points of e_i.

Since M is a connected matching in G_1^r we can find a connecting path P_i^r in G_1^r from $f^{-1}(C_2^i)$ to $f^{-1}(C_1^{i+1})$ for every $1 \leq i \leq l_1 - 1$. Note that these paths in G_1^r may not be internally vertex disjoint. From these paths P_i^r in G_1^r, we can construct vertex disjoint connecting paths P_i in G_1 connecting a typical vertex v_2^i of C_2^i to a typical vertex v_1^{i+1} of C_1^{i+1}. More precisely, we construct P_1 with the following simple greedy strategy. Denote $P_1^r = (p_1, \ldots, p_t)$, $2 \leq t \leq 3l$, where according to the definition $f(p_1) = C_2^1$ and $f(p_t) = C_1^2$. Let the first vertex u_1 $(= v_2^1)$ of P_1 be a vertex $u_1 \in C_2^1$ for which $deg_{G_1}(u_1, f(p_2)) \geq m/4$ and $deg_{G_1}(u_1, C_1^1) \geq m/4$. By ε-regularity, most of the vertices satisfy this in C_2^1. The second vertex u_2 of P_1 is a vertex $u_2 \in (f(p_2) \cap N_{G_1}(u_1))$ for which $deg_{G_1}(u_2, f(p_3)) \geq m/4$. Again by regularity most vertices satisfy this in $f(p_2) \cap N_{G_1}(u_1)$. The third vertex u_3 of P_1 is a vertex $u_3 \in (f(p_3) \cap N_{G_1}(u_2))$ for which $deg_{G_1}(u_3, f(p_4)) \geq m/4$. We continue in this fashion, finally the last vertex u_t $(= v_1^2)$ of P_1 is a vertex $u_t \in (f(p_t) \cap N_{G_1}(u_{t-1}))$ for which $deg_{G_1}(u_t, C_2^2) \geq m/4$.

Then we move on to the next connecting path P_2. Here we follow the same greedy procedure, we pick the next vertex from the next cluster in P_2^r. However, if the cluster has occurred already on the paths P_1^r or P_2^r, then we just have to make sure that we pick a vertex that has not been used on P_1 or P_2.

We continue in this fashion and construct the vertex disjoint connecting paths P_i in G_1, $1 \leq i \leq l_1 - 1$. These will be parts of the final path in G_1. We remove the internal vertices of these paths from G_1. By doing this we may create some discrepancies in the cardinalities of the clusters of this connected matching. We remove at most $(3l)^2$ vertices from each cluster of the matching to assure that now we have the same number of vertices left in each cluster of the matching. Then by applying Lemma 2 for $1 \leq i \leq l_1$, we get a path in $G_1|_{f(e_i)}$ connecting v_1^i and v_2^i that contains almost all of the vertices of $f(e_i)$ (in case of $i = 1$ we just select a long path of $f(e_1)$ starting from v_2^1 and in case of $i = l_1$, we select a long path of $f(e_{l_1})$ starting from $v_1^{l_1}$. Finally using (11) and (12) we get a path in G_1 that contains at least

$$(2 - 534\sqrt{3\varepsilon} - 16\varepsilon)n \geq (2 - 550\sqrt{3\varepsilon})n = (1 - 225\sqrt{3\varepsilon})2n = (1-\delta)2n$$

vertices. This completes the proof of Theorem 1. ■

ACKNOWLEDGEMENTS

This Research was Supported in part by OTKA grants, T038198 and T046234. and also in part by the National Science Foundation under grant no. DMS-0456401.

REFERENCES

[1] B. Bollobás, Extremal Graph Theory, Academic Press London, 1978.
[2] A. Figaj and T. Łuczak, The Ramsey number for a triple of long even cycles, J Combin Theory (to appear).
[3] L. Gerencsér and A. Gyárfás, On Ramsey-type problems, Ann Univ Sci Budapest Eötvös Sect Math 10 (1967), 167–170.
[4] R. L. Graham, B. L. Rothschild, and J. H. Spencer, Ramsey Theory, John Wiley & Sons, New York, 1990.
[5] A. Gyárfás, G. N. Sárközy, E. Szemerédi, and M. Ruszinkó, Three-color Ramsey numbers for paths, Combinatorica 27(1) (2007), 35–69.
[6] J. Komlós, G. N. Sárközy, and E. Szemerédi, Blow-up Lemma, Combinatorica 17 (1997), 109–123.
[7] J. Komlós, G. N. Sárközy, and E. Szemerédi, An algorithmic version of the Blow-up Lemma, Random Struct Algo 12 (1998), 297–312.
[8] J. Komlós and M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, Combinatorics, Paul Erdős is Eighty, D. Miklós, V. T. Sós, and T. Szőnyi (Editors), Bolyai Society Math Studies, Budapest, 1996, Vol. 2, pp. 295–352.
[9] L. Lovász and M. D. Plummer, Matching Theory, Joint edition of North-Holland and Akadémiai Kiadó, 1986.
[10] T. Łuczak, $R(C_n, C_n, C_n) \leq (4 + o(1))n$, J Combin Theory, Ser B 75 (1999), 174–187.
[11] C. Magyar and R. R. Martin, Tripartite version of the Corrádi-Hajnal theorem, Discrete Math 254 (2002), 289–308.
[12] S. P. Radziszowski, Small Ramsey numbers, Elect J Combin, (2002) DS1.
[13] F. P. Ramsey, On a problem of formal logic, Proc London Math Soc, 2nd Ser, 30 (1930), 264–286.
[14] E. Szemerédi, Regular partitions of graphs, Colloques Internationaux C.N.R.S. N$^{\underline{o}}$ 260, Problèmes Combinatoires et Théorie des Graphes, Orsay (1976), 399–401.