Consequence Relations and Natural Deduction

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1 Consequence Relations

A useful notion that cuts across both semantic (model-oriented) and syntactic (derivation-oriented) issues is the notion of a consequence relation. We will use capital Greek letters like Γ, Δ (Gamma and Delta) to refer to finite sets of formulas, and lower case Greek letters like ϕ, ψ (phi and psi) to refer to individual formulas. We will save ink by writing Γ, Δ for the set $\Gamma \cup \Delta$, and Γ, ϕ for the set $\Gamma \cup \{\phi\}$, etc.

By $\phi[t_1/x_1, \ldots t_n/x_n]$, we mean the result of plugging in the terms $t_1, \ldots, t_n$ in place of the variables $x_1, \ldots, x_n$. We assume that all the x_i are different variables, and that all of the plugging in happens at once. So, if there are x_2s inside the term t_1, they are not substituted with t_2s. $\Gamma[t_1/x_1, \ldots t_n/x_n]$ means the result of doing the substitutions to all the formulas in Γ.

Definition 1 *Suppose that $\preceq$ is a relation between finite sets of formulas and individual formulas, as in $\Gamma \preceq \phi$. Then $\preceq$ is a* consequence relation *iff it satisfies these properties:*

Reflexivity: $\Gamma, \phi \preceq \phi$;

Transitivity: $\Gamma \preceq \phi$ *and* $\Gamma, \phi \preceq \psi$ *imply* $\Gamma \preceq \psi$;

Weakening: $\Gamma \preceq \phi$ *implies* $\Gamma, \Delta \preceq \phi$*; and*

Substitution: $\Gamma \preceq \phi$ *implies* $\Gamma[t_1/x_1, \ldots t_n/x_n] \preceq \phi[t_1/x_1, \ldots t_n/x_n]$.

For now, we will focus on formulas with no variables, so **Substitution** will be irrelevant. We will ignore it until later. The **Reflexivity** and **Transitivity** rules ensure that a consequence relation is a partial order, when restricted to sets containing just one assumption. The **Weakening** rule “lifts” this partial order to sets with more members.

We refer to an instance of a relation $\Gamma \preceq \phi$ or any $\Gamma\ R\ \phi$ as a *judgment*.

Both semantic notions such as *entailment* and syntactic notions such as *derivability* give us examples of consequence relations. Suppose we have a notion of *model* such as $\mathbb{M} \models \phi$ as defined in the Dougherty lecture notes, Def. 2.2.2.[1] Then we have a corresponding notion of *(semantic) entailment* defined:

Definition 2 Γ entails ϕ*, written* $\Gamma \Vdash \phi$*, holds iff, for all models* $\mathbb{M}$*:*

If *for each* $\psi \in \Gamma$*,* $\mathbb{M} \models \psi$*,*

then $\mathbb{M} \models \phi$*.*

That is, $\Gamma \Vdash \phi$ means that every model that makes all of the formulas in Γ true makes ϕ true too.

Lemma 3 *Entailment is a consequence relation, i.e.* $\Vdash$ *satisfies reflexivity, transitivity, and weakening in Def. 1:*

1. $\Gamma, \phi \Vdash \phi$*;*
2. $\Gamma \Vdash \phi$ *and* $\Gamma, \phi \Vdash \psi$ *imply* $\Gamma \Vdash \psi$*; and*
3. $\Gamma \Vdash \phi$ *implies* $\Gamma, \Delta \Vdash \phi$*.*

We turn next to showing that a particular set of rules for constructing proofs is also a consequence relation.

[1] Available at URL `http://web.cs.wpi.edu/~guttman/cs521_website/Dougherty_lecture_notes.pdf`.

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}$$

Figure 1: ND Introduction and Elimination Rules for $\wedge$

$$\frac{\Gamma\phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \qquad \frac{\Gamma \vdash \phi \rightarrow \psi \qquad \Gamma \vdash \phi}{\Gamma \vdash \psi}$$

Figure 2: ND Introduction and Elimination Rules for $\rightarrow$

2 A Derivation System for "Natural Deduction"

We consider the rules suggested by Gerhart Gentzen as a "natural" form of deduction.[2] Gentzen considered these rules natural because they seemed to match directly the meaning of each logical operator.

Each logical operator has one or a couple of rules that allow you to prove formulas containing it as the outermost operator. These are called *introduction* rules. Each operator also has one or a couple of rules that allow you to prove other formulas by extracting the logical content in a formula containing it as outermost operator. They are called *elimination* rules. The introduction rules push formulas up in the partial ordering, while the elimination rules hold them down. Between them, the introduction and elimination rules *fix the meaning* of the logical operators purely in terms of their *deductive power*.

All of this extends to much richer logics, as we will see.

The rules are spread out through Figs. 1–4.

Definition 4 *A* natural deduction derivation *is a tree, conventionally written with the conclusion, the root, at the bottom, such that each judgment is the conclusion of a rule.*

[2]Gerhard Gentzen, "Investigations into Logical Deduction," tr. Manfred Szabo, in *Complete Works of Gerhart Gentzen*, North Holland, 1969. Originally published in *Mathematische Zeitschrift*, 1934–1935.

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}$$

$$\frac{\Gamma \vdash \phi \vee \psi \qquad \Gamma, \phi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}$$

Figure 3: ND Introduction and Elimination Rules for $\vee$

$$\frac{}{\Gamma, \phi \ \vdash \ \phi} \qquad \frac{\Gamma \ \vdash \ \bot}{\Gamma \ \vdash \ \phi}$$

Figure 4: ND Axioms and Rule for $\bot$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{p \wedge q \ \vdash \ p \wedge q}}{p \wedge q \ \vdash \ p}}{p \wedge q \ \vdash \ p \vee q}}{\vdash \ (p \wedge q) \rightarrow (p \vee q)}$$

Figure 5: An Example Derivation

A derivation is a natural deduction derivation in intuitionist propositional logic *if each rule is one of those shown in Figs. 1–4.*

An example derivation is shown in Fig. 5. It proves $\vdash \ (p \wedge q) \rightarrow (p \vee q)$. There are two questions we'd immediately like answers to. First, do the derivable judgments form a consequence relation? That is, if $\Gamma \preceq \phi$ means that there is a derivation of $\Gamma \ \vdash \ \phi$ using our rules, then is $\preceq$ a consequence relation?

Second, how do derivable judgments relate to entailment? If $\Gamma \ \vdash \ \phi$ is derivable, then is $\Gamma \Vdash \phi$ true? If $\Gamma \Vdash \phi$ then is $\Gamma \ \vdash \ \phi$ derivable?

We can answer the first question affirmatively.

Lemma 5 *The set of derivable judgments $\Gamma \ \vdash \ \phi$ form a consequence relation.*

Proof: 1. **Reflexivity** holds because $\frac{}{\Gamma, \phi \ \vdash \ \phi}$ is always a derivation.

2. **Transitivity** holds by Fig. 6.

3. Weakening holds by *induction on derivations*:

$$\cfrac{\cfrac{\begin{matrix} \vdots \\ d_1 \\ \vdots \\ \Gamma, \phi \ \vdash \ \psi \end{matrix}}{\Gamma \ \vdash \ \phi \rightarrow \psi} \qquad \begin{matrix} \vdots \\ d_2 \\ \vdots \\ \Gamma \ \vdash \ \phi \end{matrix}}{\Gamma \ \vdash \ \psi}$$

Figure 6: Composing Derivations for Transitivity

Base Case Suppose that there is a derivation of $\Gamma \vdash \phi$ consisting only of an application of the Axiom rule. That is, $\phi \in \Gamma$. Thus, $\phi \in \Gamma, \Delta$, so $\overline{\Gamma, \Delta \vdash \phi}$ is an application of the Axiom rule.

Induction Step Suppose that we are given a derivation d where the last step is an application of one of the rules from Figs. 1–4, and the previous steps generate one or more subderivations d_i, each with conclusion $\Gamma_i \vdash \psi_i$.

Induction hypothesis. Assume that for each of the subderivations d_i, there is a weakened subderivation $W(d_i)$ such that $W(d_i)$ has conclusion $\Gamma_i, \Delta \vdash \psi_i$.

Construct the desired derivation of $\Gamma, \Delta \vdash \phi$ by combining the weakened subderivations $W(d_i)$ using the *same* rule of inference.

□

One part of the second question is easy to answer.

Lemma 6 $\vdash \subseteq \Vdash$.
That is, if $\Gamma \vdash \phi$ *is derivable, then* $\Gamma \Vdash \phi$.

Proof: By induction on derivations. □

On the other hand, $\vdash \subsetneq \Vdash$. There are entailment relations that cannot be derived using these rules.

Challenge. Find a Γ, ϕ such that $\Gamma \Vdash \phi$ but $\Gamma \vdash \phi$ is not derivable using our rules. How would one prove it not derivable?

Question. If these rules do not characterize the semantic entailment relation generated from the classical $\models$, what do they characterize?