Propositional Deduction via Sequent Calculus

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Contents

A *deduction system* is a collection of formal rules that give step-by-step procedures for constructing *proofs* or *derivations*. At any stage of a derivation, there are finitely many premises or *assumptions* in force, and some *conclusion* that follows from them. The set of assumptions (as well as the conclusion) may change during the derivation: New assumptions may be added for future deduction steps, or existing assumptions can be canceled or consumed.

Suppose that a derivation has derived a formula ϕ as conclusion from the assumptions $\psi_1, \ldots, \psi_k$. This derivation is a *sound derivation* iff, for every model $\mathbb{M}$ such that $\mathbb{M} \models \{\psi_1, \ldots, \psi_k\}$, $\mathbb{M} \models \phi$. That is, it is sound iff $\{\psi_1, \ldots, \psi_k\} \Vdash \phi$.

A *deduction system* is sound if all of the derivations that can be built using it are sound derivations. Soundness is the fundamental requirement on a deduction system, and an unsound deduction system is useless.

A second desirable property for a deduction system is *completeness*. This is a kind of converse to soundness. A deduction system is complete iff, whenever $\{\psi_1, \ldots, \psi_k\} \Vdash \phi$, there is a derivation of ϕ from the premises $\psi_1, \ldots, \psi_k$.

In this note, we will introduce a derivation system for propositional logic, and prove that it is sound and complete. In the later part, we will introduce some additional rules for the quantifiers $\forall, \exists$, to produce a sound and complete system for predicate logic. Subsequently, we will also prove a kind of syntactic soundness, called consistency, which says that there is no derivation of the false formula $\bot$. The approach we take here follows Gerhard Gentzen, a German mathematician of the 1930s and early 1940s.[1]

A Note on Greek Letters. We will use Greek letters[2] such as ϕ, ψ, and χ (pronounced "phi," "psi," and "chi") to stand for individual formulas. They may be atomic or compound. For specifically atomic formulas I will continue to use Roman capital letters A, B, C, etc.

For finite sets of formulas, we will use capital Greek letters Γ and Δ (pronounced "Gamma" and "Delta"). We write Γ, ϕ for the set $\Gamma \cup \{\phi\}$, i.e. the smallest set containing all members of Γ and containing ϕ. If $\phi \in \Gamma$, then $\Gamma, \phi = \Gamma$. Σ (pronounced "Sigma") will refer to an arbitrary sequent.

1 Formulas and Sequents

For this part of the course, we will use the following definition of formulas:

Definition 1 *Let $\mathcal{L}$ be a set of values called* atomic formulas*, such that $\bot \notin \mathcal{L}$. The* formulas over *$\mathcal{L}$ are defined recursively as the smallest set containing:*

1. *each $A \in \mathcal{L}$;*

2. *the value $\bot$, called the* constantly false formula*;*

[1] Gerhard Gentzen, "Investigations into Logical Deduction," tr. Manfred Szabo, in *Complete Works of Gerhart Gentzen*, North Holland, 1969. Originally published in German, *Mathematische Zeitschrift*, 1934–1935.

[2] See the Table of Greek Letters at `http://web.cs.wpi.edu/~guttman/cs521_website/table_of_greek_letters.pdf`.

$$\Gamma, A \vdash A \qquad\qquad \Gamma, \bot \vdash A$$

Figure 1: Sequent calculus axioms

3. each of $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \to \psi$, for any formulas ϕ, ψ over $\mathcal{L}$.

The definition for $\mathbb{M} \models \phi$ stays the same, except that we add the clause that says (for all $\mathbb{M}$), that it is not the case that $\mathbb{M} \models \bot$. That is, $\bot$ is false in every model $\mathbb{M}$.

We have left out negation. We regard $\neg\phi$ as a shorthand for $\phi \to \bot$.

A *sequent* is a pair that packages together some assumptions and a conclusion. We write $\Gamma \vdash \phi$ for the sequent whose assumptions are the finite set of formulas Γ and whose conclusion is the formula ϕ. We call Γ the *antecedent* and ϕ the *consequent* of the sequent $\Gamma \vdash \phi$.

A sequent is *satisfied* in a model $\mathbb{M}$ iff either $\mathbb{M} \models \phi$ or for some $\psi \in \Gamma$, $\mathbb{M} \not\models \psi$. Later, we will consider sequents $\Gamma \vdash \Delta$ where the consequent is also a finite set, with the semantics that $\Gamma \vdash \Delta$ is *satisfied* in a model $\mathbb{M}$ iff either, for some $\phi \in \Delta$, $\mathbb{M} \models \phi$, or else, for some $\psi \in \Gamma$, $\mathbb{M} \not\models \psi$. Thus, a sequent says that if all of the formulas in its antecedent are true, then some formula in its consequent is true. I.e. an *and* of the antecedent implies an *or* of the consequent.

We write $\mathbb{M} \models \Sigma$ when sequent Σ is satisfied in model $\mathbb{M}$.

Of course, if $\bot \in \Gamma$, then $\Gamma \vdash \phi$ is automatically satisfied in any model $\mathbb{M}$, since there is some $\psi \in \Gamma$ such that $\mathbb{M} \not\models \psi$.

2 Derivation rules

Derivations are built up as trees using inferences for the different logical connectives, and axioms used as starting points. The axioms (Fig. 1) say that if an atomic formula A is assumed, then A follows, and that if the false formula $\bot$ is assumed, then any atomic formula A follows. In this latter case, we allow $\bot$ also as an instance of A.

The rules for the individual connectives are structured in a very specific way. There are either one or two rules for introducing the connective into a formula in the antecedent, and also one or two rules for introducing the connective into the consequent. The first rule in Fig. 2 says that—to prove that $\phi \wedge \psi$ follows from Γ—it's enough to prove that ϕ does, and ψ does too. Using the axioms and these rules, we can already prove the (maybe unexciting) fact that $A \wedge B \vdash A \wedge B$, as shown in Fig. 3. This derivation

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \qquad \frac{\Gamma, \phi \vdash \chi}{\Gamma, \phi \wedge \psi \vdash \chi} \qquad \frac{\Gamma, \psi \vdash \chi}{\Gamma, \phi \wedge \psi \vdash \chi}$$

Figure 2: Sequent calculus rules for conjunction

$$\frac{\dfrac{A \vdash A}{A \wedge B \vdash A} \qquad \dfrac{B \vdash B}{A \wedge B \vdash B}}{A \wedge B \vdash A \wedge B}$$

Figure 3: Derivation of $A \wedge B \vdash A \wedge B$

also illustrates the fact that even though the axioms apply only to *atomic* formulas, we will be able to build up proofs of $\phi \vdash \phi$ for all compound ϕ also. It also illustrates what a derivation is:

Definition 2 *A* derivation *is a finite tree, written with its root at the bottom, in which each node is labeled with a sequent, such that:*

1. *Each leaf is an instance of an axiom in Fig. 1, and*

2. *Each non-leaf is obtained from the one or two nodes above it by an instance of a rule from Figs. 2, 4–5.*

If d is a finite tree that satisfies Clause 2, then d is called a partial derivation.

Partial derivations are useful to represent the incomplete objects that we build in the process of searching backward (from the result) for a derivation; for now we will focus on derivations.

This definition gives us a way to prove conclusions about all *derivations* by induction. A property Prop holds of all derivations d if:

- Prop holds of each d that is an instance of an axiom (Fig. 1); and

- Suppose that derivation d is built from d_1 and d_2 by an application of one of the rules of Figs. 2, 4–5, and Prop holds of d_1 and d_2. Then Prop continues to hold of d.

This is the fundamental principle for proving results about all deductions, for instance that they all produce sound conclusions.

The rules for disjunction are *dual* to the rules for conjunction. There are two premises for the one rule for introducing $\vee$ into a formula in the antecedent, while there are two separate rules—each with one premise—to introduce $\vee$ into the consequent. The rules for implication are slightly different, because introducing an implication in the consequent actually eliminates

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi} \qquad \frac{\Gamma, \phi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \phi \vee \psi \vdash \chi}$$

Figure 4: Sequent calculus rules for disjunction

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \qquad \frac{\Gamma \vdash \phi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \phi \to \psi \vdash \chi}$$

Figure 5: Sequent calculus rules for implication

an assumption from the antecedent, as shown in Fig. 5. This is the reason why we keep track of the assumptions explicitly in each sequent. In the rule for introducing $\to$ into the antecedent, we are saying that the assumption $\phi \to \psi$ is strong enough for χ, assuming that ψ would be strong enough for χ, and assuming that the other assumptions Γ are strong enough for the hypothesis ϕ.

This rule is a bit weaker than one would like, but it is the strongest one we can write, subject to the rules of the game as we are playing it currently. These rules of the game are:

- Each rule adds a single occurrence of a connective to the conclusion;
- No rule involves any other connective besides the one it is introducing;
- Every variable appearing in a premise still appears in the conclusion;
- The consequent of each sequent consists of a single formula.

Later, we will examine some key rules of classical logic that this system does not establish—including double negation elimination $\neg(\neg\phi) \to \phi$ and excluded middle $\phi \vee \neg\phi$—and strengthen it to a system that is complete for classical logic by removing the last of these rules of the game.

3 Soundness of These Rules

In this section, we give an example of a proof by induction on the structure of derivations.

Theorem 3 (Soundness of the rules.) *If d is any derivation, with root (i.e. conclusion) $\Gamma \vdash \phi$, then $\Gamma \Vdash \phi$.*

That is, any model $\mathbb{M}$ that satisfies all of the formulas in Γ also satisfies ϕ.

The Axioms. Any instance of an axiom is surely sound: If a model $\mathbb{M}$ satisfies all of the formulas Γ, A, then it surely satisfies A. Moreover, no model $\mathbb{M}$ satisfies $\bot$, so the requirement for $\Gamma, \bot \ \vdash \ A$ is vacuously true.

The Rules. We will consider a few representative cases.

$\wedge$ **in consequent.** Suppose that d is constructed from d_1 and d_2 by combining them using $\wedge$ introduction in the consequent. Then if $\Gamma \ \vdash \ \phi$ is the root of d_1 and $\Gamma \ \vdash \ \psi$ is the root of d_2, then the conclusion of d is $\Gamma \ \vdash \ \phi \wedge \psi$. So assume $\mathbb{M} \models \Gamma$; we need to show $\mathbb{M} \models \phi \wedge \psi$.

Our induction hypothesis is that for all $\mathbb{M}'$, if $\mathbb{M}' \models \Gamma$, then $\mathbb{M}' \models \phi$, and if $\mathbb{M}' \models \Gamma$, then $\mathbb{M}' \models \psi$.

Applying the induction hypothesis to $\mathbb{M}$, we infer that $\mathbb{M} \models \phi$ and $\mathbb{M} \models \psi$. By the definition of $\models$, $\mathbb{M} \models \phi \wedge \psi$.

$\wedge$ **in antecedent.** Suppose that d is constructed from d_1 by $\wedge$ introduction in the antecedent. Then if $\Gamma, \phi \ \vdash \ \chi$ is the root of d_1, then the conclusion of d is of the form $\Gamma, \phi \wedge \psi \ \vdash \ \chi$. So assume $\mathbb{M} \models \Gamma, \phi \wedge \psi$; we need to show $\mathbb{M} \models \chi$.

Our induction hypothesis is that for all $\mathbb{M}'$, if $\mathbb{M}' \models \Gamma, \phi$, then $\mathbb{M}' \models \chi$.

Since $\mathbb{M} \models \Gamma, \phi \wedge \psi$, we know that $\mathbb{M} \models \Gamma, \phi$. Applying the induction hypothesis to $\mathbb{M}$, it follows that $\mathbb{M} \models \chi$.

$\rightarrow$ **in antecedent.** Suppose that d is constructed from d_1 and d_2 by combining them using $\rightarrow$ introduction in the antecedent. Then if $\Gamma \ \vdash \ \phi$ is the root of d_1 and $\Gamma, \psi \ \vdash \ \chi$ is the root of d_2, then the conclusion of d is $\Gamma, \phi \rightarrow \psi \ \vdash \ \chi$. So assume $\mathbb{M} \models \Gamma, \phi \rightarrow \psi$; we need to show $\mathbb{M} \models \chi$.

Our induction hypothesis is that for all $\mathbb{M}'$, if $\mathbb{M}' \models \Gamma$, then $\mathbb{M}' \models \phi$, and if $\mathbb{M}' \models \Gamma, \psi$, then $\mathbb{M}' \models \chi$.

There are two cases if $\mathbb{M} \models \Gamma, \phi \rightarrow \psi$:

(i) $\mathbb{M} \models \Gamma, \neg\phi$, and
(ii) $\mathbb{M} \models \Gamma, \psi$.

In case (ii), the second part of the induction hypothesis yields the conclusion $\mathbb{M} \models \chi$. Moreover, the first part of the induction hypothesis implies that case (i) does not occur. $\square$

Exercise 4 *Check the remaining cases in the induction to complete the proof of the soundness of the rules.*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, A \to B \vdash B}}{A \vdash (A \to B) \to B} \qquad B \vdash B}{A, ((A \to B) \to B) \to B \vdash B}}{((A \to B) \to B) \to B \vdash A \to B}}{\vdash (((A \to B) \to B) \to B) \to (A \to B)}}{}
$$

Figure 6: Proof of triple negation elimination

$$
\dfrac{\dfrac{\dfrac{A \vdash \bot}{\vdash A \to \bot} \qquad A \vdash A}{(A \to \bot) \to \bot \vdash A}}{\vdash ((A \to \bot) \to \bot) \to A}
$$

Figure 7: Fragment attempting to justify double negation elimination

4 Use of the Rules; Their Limits

These rules are easy to use. We illustrate this by proving that $\neg\neg\neg A \to \neg A$. I.e. when more than two negations are present, pairs of negations may be discarded. Since $\neg\phi$ abbreviates $\phi \to \bot$, this goal formula is actually

$$(((A \to \bot) \to \bot) \to \bot) \to (A \to \bot).$$

The same derivation also works for the more general

$$(((A \to B) \to B) \to B) \to (A \to B),$$

so we actually prove the latter. We show the derivation in Fig. 6. However, if we try to construct a similar proof of the (classically valid) formula for double negation elimination, $((A \to \bot) \to \bot) \to A$, working our way up from the bottom, we obtain the fragment shown in Fig. 7, which cannot be completed to a derivation. Thus, this inference system is not complete, relative to the classical (model-based) semantics.

Intuitionistic Logic and Constructive Reasoning. The deduction system of this section is incomplete for classical logic, but it is correct for *intuitionistic logic*.[3] Intuitionism, initiated by the great Dutch topologist L. E. J. Brouwer, was an important component of work in the foundations

[3] See e.g. Dirk van Dalen, "Intuitionistic Logic," in *The Blackwell Guide to Philosophical Logic*, ed. L. Gobble. Blackwell, Oxford. 2001, 224–257. Available at `http://www.phil.uu.nl/~dvdalen/articles/Blackwell(Dalen).pdf`.

of mathematics from early in the 20th century, and it has played a major role in theoretical computer science in the past twenty or thirty years.

The intuitionist (or more generally, "constructivist") point of view is that mathematics is not about abstract reasoning, but about mathematical computations and other *constructions*. In geometry, we know what the constructions are, and in arithmetic (e.g.) we know what the computations are. When this idea is applied to logic, the relevant constructions are *proofs*, and the meaning of a formula is given—not in terms of models as we are doing in this course—but in terms of what is needed to construct a proof of that formula.

From this point of view, to demonstrate a sequent $\Gamma \ \vdash \ \phi$, we should provide a recipe which, when supplied with proofs of the formulas in Γ, is capable of transforming them into a proof of the consequent ϕ. The inference system of Figs. 1–2, 4–5 is correct relative to this point of view. The rule for conjunction in the consequent, for instance, provides a way to produce a proof of $\phi \wedge \psi$ by piecing together the embedded proofs of ϕ and ψ furnished in the premises. The rules for disjunction in the consequent provide a way to prove $\phi \vee \psi$ by identifying one of the disjuncts, and providing a proof of that one, namely the proof furnished by the premise.

From this point of view, the most interesting rule is the one for implication in the antecedent. The intuitionist view is that a proof of an implication $\phi \to \psi$ ought to be a *function*, which when supplied with a proof of the hypothesis ϕ will provide a proof of the conclusion ψ. The rule

$$\frac{\Gamma \ \vdash \ \phi \qquad \Gamma, \psi \ \vdash \ \chi}{\Gamma, \phi \to \psi \ \vdash \ \chi}$$

provides a way to piece together recipes. Suppose we already have recipes (i) to construct a proof of ϕ from proofs of the formulas Γ, and (ii) to construct a proof of χ from proofs of the formulas Γ, ψ.

We want a recipe to construct a proof of χ, given proofs of $\Gamma, \phi \to \psi$. The rule is correct, because we can act as follows:

- Use recipe (i) to construct a proof d_ϕ of ϕ, using the proofs of Γ;
- The proof of $\phi \to \psi$ is a function f from proofs of ϕ to proofs of ψ, so apply f to d_ϕ to obtain a proof d_ψ of ψ;
- Apply recipe (ii) to d_ψ in combination with the given proofs of Γ, thereby obtaining the desired proof of χ.

From this point of view, there is no reason why we should expect $((A \to \bot) \to \bot) \to A$. The formula $\bot$ is the formula that can never be proved.

Thus, a proof of $A \to \perp$ would be a function that—if it would ever be given a proof of A—would have to return something that does not exist. Thus, $A \to \perp$ says that there are no proofs of A. $(A \to \perp) \to \perp$ thus says that there are no proofs *that A has no proofs*. How would we construct a function f to transform a proof d that there are no proofs that A has no proofs into a construction of A?

There presumably is no such transformation in general. Of course, if we know a proof d_A of A, then we could use the transformation that ignores d and returns d_A. There are other situations in which we could produce transformations (sometimes less trivial ones), but the principle of double negation elimination does not have an intuitionistic justification in general.

Similarly, there is no reason why we should have a proof of $\phi \vee \neg\phi$ regardless of the content of ϕ. Such a proof d would have to identify one of the disjuncts, either ϕ or $\neg\phi$, and provide a proof of that disjunct. But of course this requires d to "know which disjunct to prove," which is impossible in general.

Intuitionist logics are very natural for providing semantics of programming languages and models of computation; see e.g. Benjamin C. Pierce, *Types and Programming Languages*. Cambridge: MIT Press, 2002. We now return you to your regularly scheduled program on *classical* logic.

5 A Complete Classical Deduction System

To obtain a classically complete deduction system, we take just one simple step. We change the definition of *sequent* to allow a finite number of formulas in the consequent. The interpretation is that if *all* of the formulas in the antecedent are true, then *at least one* formula in the consequent is true. That is, in a sequent Σ of the form $\Gamma \vdash \Delta$, the *conjunction* of all formulas in Γ implies the *disjunction* of all formulas in Δ. We write $\mathbb{M} \models \Sigma$ when sequent Σ is satisfied in model $\mathbb{M}$ in this sense.

We then write the full deduction system in a form that allows formulas Δ to be carried along in the consequent (see Fig. 8). These changes are mechanical, but we present the full deduction system again, just so it is clear how to write all the rules with multiple formulas in the consequent. We refer to the sets Γ, Δ in these rules as the *sideformulas*, since they are simply carried alongside the main action as the inference occurs.

One essential change is to the rule for implication introduction in the antecedent. In effect, it allows us to strengthen the rule by carrying along χ as we pass up the derivation on the left branch; we show here the old rule,

$$\Gamma, A \vdash A, \Delta \qquad\qquad \Gamma, \bot \vdash A, \Delta$$

$$\frac{\Gamma \vdash \phi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta} \qquad \frac{\Gamma, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \qquad \frac{\Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \qquad \frac{\Gamma, \phi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \to \psi, \Delta} \qquad \frac{\Gamma \vdash \phi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \to \psi \vdash \Delta}$$

Figure 8: A System of Axioms and Rules for Classical Deduction

$$\cfrac{\cfrac{\cfrac{A \vdash \bot, A}{\vdash A \to \bot, A} \qquad \bot \vdash A}{(A \to \bot) \to \bot \vdash A}}{\vdash ((A \to \bot) \to \bot) \to A} \qquad\qquad \cfrac{\cfrac{\cfrac{A \vdash \bot, A}{\vdash A \to \bot, A}}{\vdash A \vee (A \to \bot), A}}{\vdash A \vee (A \to \bot)}$$

Figure 9: Double negation elimination and excluded middle

and the new rule in the case in which Δ is the single formula χ:

$$\frac{\Gamma \vdash \phi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \phi \to \psi \vdash \chi} \quad \mapsto \quad \frac{\Gamma \vdash \phi, \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \phi \to \psi \vdash \chi}$$

We can use this new version of the rule to construct proofs of double-negation elimination and of the law of the excluded middle (Fig. 9). The proof of excluded middle looks a little peculiar. The last two steps are both instances of $\vee$-introduction in the consequent. The next-to-last step replaces $A \to \bot$ by $A \vee (A \to \bot)$, and its sideformulas Δ are the singleton set $\Delta = \{A\}$. The last step replaces A by $A \vee (A \to \bot)$, with $\Delta = \{A \vee (A \to \bot)\}$. The resulting sequent has $A \vee (A \to \bot)$ "twice," so to speak, but since its consequent is a set, that is the same as having it once.

This ability to use a disjunction twice allows us (from the bottom-up perspective of proof construction) to extract both disjuncts. It suggests that we can replace the two rules for $\vee$-introduction in the consequent by the following distinctly classical rule. Symmetrically, we replace the two rules for $\wedge$-introduction in the antecedent by one rule:

$$\frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \qquad \frac{\Gamma, \phi, \psi \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$$

These forms are particularly convenient for proof search. We write CSC for the deduction system that contains these two rules, together with the

$$\Gamma, A \vdash A, \Delta \qquad\qquad \Gamma, \bot \vdash A, \Delta$$

$$\frac{\Gamma \vdash \phi, \Delta \qquad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \phi \wedge \psi, \Delta} \qquad\qquad \frac{\Gamma, \phi, \psi, \vdash \Delta}{\Gamma, \phi \wedge \psi \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \psi, \Delta}{\Gamma \vdash \phi \vee \psi, \Delta} \qquad\qquad \frac{\Gamma, \phi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \vee \psi \vdash \Delta}$$

$$\frac{\Gamma, \phi \vdash \psi, \Delta}{\Gamma \vdash \phi \rightarrow \psi, \Delta} \qquad\qquad \frac{\Gamma \vdash \phi, \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, \phi \rightarrow \psi \vdash \Delta}$$

Figure 10: The System CSC of Axioms and Rules for Classical Deduction

axioms and rules of Fig. 8 other than $\vee$-introduction in the consequent and $\wedge$-introduction in the antecedent. The system CSC is summarized in Fig. 10.

6 Proof Search

Suppose we have a sequent $\Gamma_0 \vdash \Delta_0$ that we would like to prove, if it is valid. How can we systematically construct progressively larger partial derivations, so that if there is a derivation of $\Gamma_0 \vdash \Delta_0$, we will find it? This is called the *proof search* problem, and it has a particularly simple answer in our deduction system for propositional logic.

At any stage in a proof search, we have a partial derivation, which is initially a tree containing only the root $\Gamma_0 \vdash \Delta_0$. The process proceeds according to the following rules:

- If every topmost node (leaf) is an instance of the axioms, then the partial derivation is in fact a derivation, and the proof search has *terminated successfully.*

- If some leaf contains no logical connectives, but is not an instance of the axioms, then the search has *failed*, i.e. terminated unsuccessfully.

- If the search has not terminated, then there is some leaf $\Gamma \vdash \Delta$ that is not an instance of an axiom, and contains a connective. A search step may choose the main connective of any compound formula of the antecedent or consequent. The choice—selecting antecedent or consequent, and selecting one formula and its main connective—determines one applicable rule from CSC.

This process must terminate, since every step introduces premises with one fewer connective than the total number in the leaf chosen to operate on. Thus, no branch can be of length greater than the total number of connectives in the root $\Gamma_0 \vdash \Delta_0$.

Proof search is non-deterministic, since—when a sequent contains several compound formulas—one may choose any one of them to break down immediately. However, we will prove that no choice can be a bad choice: If any run of the proof search process leads to a successfully terminated result, then all runs do. Of course, some runs could be much longer than others, but the outcome will be the same.

7 Completeness Theorem

The essential fact about the rules of CSC is the preservation of models.

Lemma 5 *1. If a sequent $\Gamma \vdash \Delta$ is an instance of an axiom, then for all models $\mathbb{M}$, $\mathbb{M} \models \Gamma \vdash \Delta$.*

2. If a sequent $\Gamma \vdash \Delta$ is not an instance of an axiom, but Γ, Δ contain only atomic formulas, then there is a model $\mathbb{M}$ such that $\mathbb{M} \not\models \Gamma \vdash \Delta$.

3. Suppose that

$$R_1 = \frac{\Sigma_1}{\Sigma_2} \qquad \textit{or} \qquad R_2 = \frac{\Sigma_0 \qquad \Sigma_1}{\Sigma_2}$$

is an instance of any rule of CSC, and $\mathbb{M}$ is a model. For R_1, $\mathbb{M} \models \Sigma_2$ iff $\mathbb{M} \models \Sigma_1$. For R_2, $\mathbb{M} \models \Sigma_2$ iff both $\mathbb{M} \models \Sigma_0$ and $\mathbb{M} \models \Sigma_1$.

We already proved item 1 in the proof of Theorem 3. For item 2, let $\mathbb{M}$ assign an atomic formula A to be true if it is a member of Γ and false if it is a member of Δ. This is a consistent assignment because $\Gamma \cap \Delta = \emptyset$, and because $\perp \notin \Gamma$. By the definition, $\mathbb{M} \not\models \Gamma \vdash \Delta$.

As for item 3, first consider all of the rules together. If a model $\mathbb{M} \not\models \Gamma$, or if $\mathbb{M} \models \rho$ for any $\rho \in \Delta$, then $\mathbb{M}$ satisfies the premises and also the conclusion of all the rules.

Thus, in considering each rule, we consider only the case in which $\mathbb{M} \models \Gamma$ and $\mathbb{M} \not\models \rho$ for each $\rho \in \Delta$. We show here four cases.

$\vee$ **in antecedent.** The conclusion is satisfied iff $\mathbb{M} \models \phi \vee \psi$, and the premise is satisfied iff $\mathbb{M} \models \phi$ or $\mathbb{M} \models \psi$, which is equivalent.

$\wedge$ **in consequent.** $\mathbb{M} \models \phi \wedge \psi$ iff $\mathbb{M} \models \phi$ and $\mathbb{M} \models \psi$, as desired.

$\wedge$ **in antecedent.** Again, our equivalence requires that $\mathbb{M} \models \phi \wedge \psi$ iff $\mathbb{M} \models \phi$ and $\mathbb{M} \models \psi$, as desired.

$\rightarrow$ **in antecedent.** The conclusion is satisfied iff $\mathbb{M} \not\models \phi \rightarrow \psi$, i.e. iff $\mathbb{M} \models \phi$ and $\mathbb{M} \not\models \psi$. The first premise is satisfied iff $\mathbb{M} \models \phi$, and the second premise is satisfied iff $\mathbb{M} \not\models \psi$. Thus, both premises are satisfied iff the conclusion is. $\square$

Exercise 6 *Check the remaining cases in the induction to complete the proof of Lemma 5.*

We can now prove:

Theorem 7 (Completeness of CSC) *Let d be a (possibly) partial derivation, produced by a terminated proof search, with root Σ.*

1. *For every model $\mathbb{M}$, $\mathbb{M}$ satisfies Σ iff for every leaf sequent Σ' in d, $\mathbb{M}$ satisfies Σ'.*

2. *Σ is a valid sequent iff every leaf sequent of d is an instance of an axiom; equivalently, iff d is a derivation.*

3. *If d' is also a (possibly) partial derivation, produced by a terminated proof search, with the same root Σ, then d is a derivation iff d' is a derivation.*

Proof. Clause 1 follows by induction, using Lemma 5, Clause 3. Clause 2 follows using Lemma 5, Clauses 1 and 2. Finally, applying Clause 2 twice, d is a derivation iff Σ is valid, which holds iff d' is a derivation. $\square$

8 Predicate Logic

To extend the system CSC of rules for classical propositional logic to a system of rules for predicate logic (shown in Fig. 11), we only need to add a pair of rules for $\forall$ and a pair of rules for $\exists$. Each operator gets a pair of rules with one rule adding it on each side of the sequent sign $\vdash$. Appealingly, the duality we observed in $\wedge$ and $\vee$ is equally visible in $\forall$ and $\exists$.

We use the convention that these rules can be applied with any variable in the quantified position here marked by x. In these rules, we use the notation $\phi[t/x]$, which means t inserted in place of all free occurrences of x,

$$\frac{\Gamma \vdash \phi[t/x], \Delta}{\Gamma \vdash \exists x \,.\, \phi, \Delta} \qquad * \frac{\Gamma, \phi \vdash \Delta}{\Gamma, \exists z \,.\, \phi[z/x] \vdash \Delta}$$

$$\frac{\Gamma \vdash \phi, \Delta}{\Gamma \vdash \forall z \,.\, \phi[z/x], \Delta} * \qquad \frac{\Gamma, \phi[t/x] \vdash \Delta}{\Gamma, \forall x \,.\, \phi \vdash \Delta}$$

Figure 11: Rules for Quantifiers. Side condition $(*)$: $x \notin \mathsf{fv}(\Gamma, \Delta)$ and either x is the same variable as z, or else $z \notin \mathsf{fv}(\phi)$

on the assumption that no variable free in t becomes bound as a consequence of the substitution:

Definition 8 (Capture-avoiding substitution) *Suppose that t is a term and ϕ is a formula with 0 or more free occurrences of the variable x.*

1. *If a variable $v \in \mathsf{fv}(t)$ is bound at a location*[4] *in ϕ at which x occurs free, then we say that t is* captured *for x in ϕ at this location.*
2. *If t is not captured for x at any location in ϕ, then $\phi[t/x]$ is well-defined, and is the result of replacing each free occurrence of x in ϕ by the term t. Otherwise, $\phi[t/x]$ is not well-defined.*
3. *The* capture-avoiding result of substituting *t for x means $\phi[t/x]$ when the latter is defined.*

There is one additional complicating factor. Whether one can apply the $\vdash \ \forall$ rule or the $\exists \ \vdash$ rule depends on which variables occur free in the remaining formulas. In particular, the variable of quantification (x here) must not occur free in the remaining formulas Γ, Δ. In the case of $\vdash \ \forall$, this amounts to saying that $\forall x \,.\, \phi$ holds when ϕ is proved with a free variable x about which nothing is assumed. The fact that x simply does not occur free in Γ, Δ means that those formulas cannot possibly make any assumption or place any constraint on its value.

For $\exists \ \vdash$ the concept is similar, except dual. That is, suppose we can infer that some formula in Δ holds, assuming Γ and that ϕ holds. If we have assumed something about x, it may not be enough just to assume $\Gamma, \exists x \,.\, \phi$. We may need some specific kind of x to satisfy ϕ, not just any x. But if x does not occur free in Γ, Δ, then any x is as good as another, and Δ will hold assuming $\Gamma, \exists x \,.\, \phi$.

[4] This means that there is a quantifier $\forall v$ or $\exists v$ above this location, on the path leading to the root of the formula (regarded as a tree).

For this reason, we use the *side condition* that $x \notin \mathsf{fv}(\Gamma, \Delta)$ to restrict the application of $\vdash \forall$ and $\exists \vdash$ to the cases where x is really "syntactically general" in Γ, Δ.