Natural Deduction: Making Proofs Explicit

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Today's Goal

Proofs are computational objects.
We want to show their computational content
explicitly in our formal system.

Conjunction Rules

$$\frac{\Gamma \ \vdash \ \varphi \qquad \Gamma \ \vdash \ \psi}{\Gamma \ \vdash \ \varphi \wedge \psi}$$

$$\frac{\Gamma \ \vdash \ \varphi \wedge \psi}{\Gamma \ \vdash \ \varphi} \qquad\qquad \frac{\Gamma \ \vdash \ \varphi \wedge \psi}{\Gamma \ \vdash \ \psi}$$

Conjunction Rules

$$\frac{\Gamma \vdash s : \varphi \qquad \Gamma \vdash t : \psi}{\Gamma \vdash \langle s, t \rangle : \varphi \wedge \psi}$$

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$$\frac{\Gamma \vdash s : \varphi \wedge \psi}{\Gamma \vdash \mathsf{fst}(s) : \varphi} \qquad\qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

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Axiom and Bottom

$$\frac{}{\Gamma, \varphi \ \vdash \ \varphi} \qquad \frac{\Gamma \ \vdash \ \bot}{\Gamma \ \vdash \ \varphi}$$

Axiom and Bottom

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

A Derivation

$$\dfrac{\dfrac{\overline{p \wedge q \;\vdash\; p \wedge q}}{p \wedge q \;\vdash\; p}}{\vdash\; (p \wedge q) \rightarrow p}$$

A Derivation

$$
\cfrac{\cfrac{}{x\colon p \wedge q \;\vdash\; x\colon p \wedge q}}{\cfrac{p \wedge q \;\vdash\; p}{\vdash\; (p \wedge q) \to p}}
$$

A Derivation

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A Derivation

$$\dfrac{\dfrac{\overline{x\colon p \wedge q \;\vdash\; x\colon p \wedge q}}{x\colon p \wedge q \;\vdash\; \mathsf{fst}(x)\colon p}}{\vdash\; \text{function } x \mapsto \mathsf{fst}(x)\colon (p \wedge q) \rightarrow p}$$

A Derivation

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We write this function:

$$\lambda x \,.\, \mathsf{fst}(x)$$

Lambda and Functions

- What's the derivative of $3xy^2$?

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Lambda and Functions

- What's the derivative of $3xy^2$? $D(\lambda y\ .\ 3xy^2)$
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$$D(\lambda y\ .\ 3xy^2)$$
$$= \lambda y\ .\ 6xy$$

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  - With respect to x, its derivative is $3y^2$ $= \lambda x\ .\ 3y^2$
  - So if $y = 3$, that's the function whose value is always 27 $= \lambda x\ .\ 27$

A Derivation

$$\cfrac{\cfrac{\overline{x : p \wedge q \ \vdash \ x : p \wedge q}}{x : p \wedge q \ \vdash \ \mathsf{fst}(x) : p}}{\vdash \ \lambda x \,.\, \mathsf{fst}(x) : (p \wedge q) \rightarrow p}$$

Implication Rules

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}$$

$$\frac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

Implication Rules

$$\frac{\Gamma, x\colon \varphi \vdash s\colon \psi}{\Gamma \vdash \lambda x\,.\,s\colon \varphi \to \psi}$$

$$\frac{\Gamma \vdash \varphi \to \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

Implication Rules

$$\frac{\Gamma, x\colon \varphi \;\vdash\; s\colon \psi}{\Gamma \;\vdash\; \lambda x\,.\,s\colon \varphi \to \psi}$$

$$\frac{\Gamma \;\vdash\; s\colon \varphi \to \psi \qquad \Gamma \;\vdash\; t\colon \varphi}{\Gamma \;\vdash\; (st)\colon \psi}$$

A Proof of an Implication is a Function

from proofs of the hypothesis
to proofs of the conclusion

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ d_1\\ \vdots\\ \Gamma, \varphi \vdash \psi\end{array}}{\Gamma \vdash \varphi \to \psi} \qquad \begin{array}{c}\vdots\\ d_2\\ \vdots\\ \Gamma \vdash \varphi\end{array}}{\Gamma \vdash \psi}$$

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\dfrac{\dfrac{\begin{array}{c}\vdots\\ d_1\\ \vdots\\ \Gamma, x\colon \varphi \vdash s\colon \psi\end{array}}{\Gamma \vdash \lambda x\,.\,s\colon \varphi \to \psi} \qquad \begin{array}{c}\vdots\\ d_2\\ \vdots\\ \Gamma \vdash \varphi\end{array}}{\Gamma \vdash \psi}
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\dfrac{\dfrac{\begin{array}{c}\vdots \\ d_1 \\ \vdots \\ \Gamma, x : \varphi \;\vdash\; s : \psi\end{array}}{\Gamma \;\vdash\; \lambda x \,.\, s : \varphi \to \psi} \qquad \begin{array}{c}\vdots \\ d_2 \\ \vdots \\ \Gamma \;\vdash\; t : \varphi\end{array}}{\Gamma \;\vdash\; (\lambda x \,.\, s) t : \psi}
$$

A Proof of an Implication is a Function

from proofs of the hypothesis
to proofs of the conclusion

$$
\dfrac{\dfrac{\begin{matrix} \vdots \\ d_1 \\ \vdots \\ \Gamma, x\colon \varphi \vdash s\colon \psi \end{matrix}}{\Gamma \vdash \lambda x\,.\, s\colon \varphi \to \psi} \qquad \begin{matrix} \vdots \\ d_2 \\ \vdots \\ \Gamma \vdash t\colon \varphi \end{matrix}}{\Gamma \vdash s[t/x]\colon \psi}
$$

Disjunction Rules

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$\frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}$$

Disjunction Rules

$$\frac{\Gamma \vdash s : \varphi}{\Gamma \vdash \langle \mathsf{lft}, s \rangle : \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$\frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}$$

Disjunction Rules

$$\frac{\Gamma \vdash s : \varphi}{\Gamma \vdash \langle \mathsf{lft}, s \rangle : \varphi \vee \psi} \qquad \frac{\Gamma \vdash s : \psi}{\Gamma \vdash \langle \mathsf{rgt}, s \rangle : \varphi \vee \psi}$$

$$\frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}$$

Disjunction Rules

$$\frac{\Gamma \vdash s : \varphi}{\Gamma \vdash \langle \mathsf{lft}, s \rangle : \varphi \vee \psi} \qquad \frac{\Gamma \vdash s : \psi}{\Gamma \vdash \langle \mathsf{rgt}, s \rangle : \varphi \vee \psi}$$

$$\frac{\Gamma \vdash s : \varphi \vee \psi \qquad \Gamma, x : \varphi \vdash t : \chi \qquad \Gamma, y : \psi \vdash r : \chi}{\Gamma \vdash \mathsf{cases}(s, \lambda x\, .\, t, \lambda y\, .\, r) : \chi}$$

Another Derivation

$$\dfrac{\dfrac{\dfrac{}{p \wedge q \;\vdash\; p \wedge q}}{p \wedge q \;\vdash\; p}}{\dfrac{p \wedge q \;\vdash\; p \vee q}{(p \wedge q) \rightarrow (p \vee q)}}$$

Another Derivation

$$\cfrac{\cfrac{\cfrac{\overline{x : p \land q \;\vdash\; x : p \land q}}{p \land q \;\vdash\; p}}{p \land q \;\vdash\; p \lor q}}{(p \land q) \to (p \lor q)}$$

Another Derivation

$$\cfrac{\cfrac{\cfrac{}{x\colon p \wedge q \ \vdash \ x\colon p \wedge q}}{x\colon p \wedge q \ \vdash \ \mathsf{fst}(x)\colon p}}{\cfrac{p \wedge q \ \vdash \ p \vee q}{(p \wedge q) \rightarrow (p \vee q)}}$$

Another Derivation

$$\dfrac{\dfrac{\dfrac{}{x : p \wedge q \ \vdash \ x : p \wedge q}}{x : p \wedge q \ \vdash \ \mathsf{fst}(x) : p}}{\dfrac{x : p \wedge q \ \vdash \ \langle \mathsf{lft}, \mathsf{fst}(x) \rangle : p \vee q}{(p \wedge q) \rightarrow (p \vee q)}}$$

Another Derivation

$$
\dfrac{\dfrac{\dfrac{\overline{x : p \wedge q \vdash x : p \wedge q}}{x : p \wedge q \vdash \mathsf{fst}(x) : p}}{x : p \wedge q \vdash \langle \mathsf{lft}, \mathsf{fst}(x) \rangle : p \vee q}}{\lambda x \,.\, \langle \mathsf{lft}, \mathsf{fst}(x) \rangle : (p \wedge q) \rightarrow (p \vee q)}
$$

Axiom and Bottom

$$\frac{}{\Gamma, x : \varphi \vdash x : \varphi} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

Axiom and Bottom

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Axiom and Bottom

$$\frac{}{\Gamma, x \colon \varphi \vdash x \colon \varphi}$$

$$\frac{\Gamma \vdash x \colon \bot}{\Gamma \vdash \mathrm{emp}(x) \colon \varphi}$$

Axiom and Bottom

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The Empty Promise Rule

Axiom and Bottom

$$\frac{}{\Gamma, x\colon \varphi \ \vdash \ x\colon \varphi} \qquad\qquad \frac{\Gamma \ \vdash \ x\colon \bot}{\Gamma \ \vdash \ \mathsf{emp}(x)\colon \varphi}$$

The Wishful Thinking Rule

Wishful Thinking

What about $\mathrm{fst}(x)$?

$$
\cfrac{\cfrac{\cfrac{\overline{x : p \wedge q \;\vdash\; x : p \wedge q}}{x : p \wedge q \;\vdash\; \mathrm{fst}(x) : p}}{x : p \wedge q \;\vdash\; \langle \mathrm{lft}, \mathrm{fst}(x) \rangle : p \vee q}}{\lambda x \,.\, \langle \mathrm{lft}, \mathrm{fst}(x) \rangle : (p \wedge q) \rightarrow (p \vee q)}
$$

Wishful Thinking

What about fst(x)?

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x\colon p \wedge q \;\vdash\; x\colon p \wedge q}}{x\colon p \wedge q \;\vdash\; \mathrm{fst}(x)\colon p}}{x\colon p \wedge q \;\vdash\; \langle \mathrm{lft}, \mathrm{fst}(x)\rangle\colon p \vee q}}{\lambda x\;.\;\langle \mathrm{lft}, \mathrm{fst}(x)\rangle\colon (p \wedge q) \rightarrow (p \vee q)}$$

This is like typechecking

Propositions as types

- A proposition is a type
- A proof ("construction") is an object of that type
- The rules must ensure:

If	a value s can be bound to a variable x
and	x occurs in an elimination context, e.g. $\mathrm{fst}(x)$
then	s permits that operation, e.g. $s = \langle t_1, t_2 \rangle$

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- A proposition is a type
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  and x occurs in an elimination context, e.g. $\mathrm{fst}(x)$
  then s permits that operation, e.g. $s = \langle t_1, t_2 \rangle$

  What are
  - *the computational rules for manipulating these objects?*
  - *the properties ensured by those rules?*