The Simply Typed Lambda Calculus λ_{ipl}: Computing with Explicit Proofs

Joshua D. Guttman

Worcester Polytechnic Institute

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Proof objects of λ_{ipl}

$$
\begin{array}{lllllll}
s & ::= & v & \mid & & & \\
 & & \langle s, s' \rangle & \mid & \text{fst}(s) & \mid & \text{scd}(s) & \mid \\
 & & (\lambda v \,.\, s) & \mid & s\ s' & \mid & & \\
 & & \langle \text{lft}, s \rangle & \mid & \langle \text{rgt}, s \rangle & \mid & \text{cases}(s, t, r) & \mid \\
v & ::= & x & \mid & y & \mid & v' & \mid & \ldots
\end{array}
$$

Reduction rules for λ_{ipl}

$$\mathsf{fst}(\langle s, s' \rangle) \longrightarrow s$$

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$$\begin{array}{lcl} \mathsf{fst}(\langle s, s' \rangle) & \longrightarrow & s \\ \mathsf{scd}(\langle s, s' \rangle) & \longrightarrow & s' \end{array}$$

Reduction rules for λ_{ipl}

$$
\begin{array}{lcl}
\mathrm{fst}(\langle s, s' \rangle) & \longrightarrow & s \\
\mathrm{scd}(\langle s, s' \rangle) & \longrightarrow & s' \\
\mathrm{cases}(\langle \mathrm{lft}, s \rangle, t, r) & \longrightarrow & t\ s
\end{array}
$$

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\mathrm{fst}(\langle s, s' \rangle) & \longrightarrow & s \\
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\mathrm{cases}(\langle \mathrm{lft}, s \rangle, t, r) & \longrightarrow & t\ s \\
\mathrm{cases}(\langle \mathrm{rgt}, s \rangle, t, r) & \longrightarrow & r\ s
\end{array}
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\begin{array}{lcl}
\mathrm{fst}(\langle s, s' \rangle) & \longrightarrow & s \\
\mathrm{scd}(\langle s, s' \rangle) & \longrightarrow & s' \\
\mathrm{cases}(\langle \mathrm{lft}, s \rangle, t, r) & \longrightarrow & t\ s \\
\mathrm{cases}(\langle \mathrm{rgt}, s \rangle, t, r) & \longrightarrow & r\ s \\
(\lambda v\ .\ s)s' & \longrightarrow & s[s'/v]
\end{array}
$$

Reduction and Proofs: $\quad \mathsf{fst}(\langle s, t\rangle) \longrightarrow s$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma \vdash \varphi\end{array} \qquad \begin{array}{c}\vdots \\ t \\ \vdots \\ \Gamma \vdash \psi\end{array}}{\Gamma \vdash \varphi \wedge \psi}}{\Gamma \vdash \varphi}
$$

Reduction and Proofs: $\mathrm{fst}(\langle s, t\rangle) \longrightarrow s$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ s\\ \vdots\\ \Gamma \vdash s : \varphi\end{array} \qquad \begin{array}{c}\vdots\\ t\\ \vdots\\ \Gamma \vdash t : \psi\end{array}}{\Gamma \vdash \langle s, t\rangle : \varphi \wedge \psi}}{\Gamma \vdash \varphi}$$

Reduction and Proofs: $\mathsf{fst}(\langle s, t\rangle) \longrightarrow s$

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Reduction and Proofs: $\text{fst}(\langle s, t \rangle) \longrightarrow s$

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Reduction and Proofs: $\mathrm{fst}(\langle s, t\rangle) \longrightarrow s$

$$\begin{array}{c} \vdots \\ s \\ \vdots \\ \Gamma \vdash s : \varphi \end{array}$$

Reduction and Proofs: $\mathrm{scd}(\langle s, t \rangle) \longrightarrow t$

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma \vdash \varphi\end{array} \qquad \begin{array}{c}\vdots \\ t \\ \vdots \\ \Gamma \vdash \psi\end{array}}{\Gamma \vdash \varphi \wedge \psi}}{\Gamma \vdash \psi}$$

Reduction and Proofs: $\mathrm{scd}(\langle s, t\rangle) \longrightarrow t$

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Reduction and Proofs: $\mathrm{scd}(\langle s, t\rangle) \longrightarrow t$

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma \vdash s : \varphi\end{array} \qquad \begin{array}{c}\vdots \\ t \\ \vdots \\ \Gamma \vdash t : \psi\end{array}}{\Gamma \vdash \langle s, t\rangle : \varphi \wedge \psi}}{\Gamma \vdash \mathrm{scd}(\langle s, t\rangle) : \psi}$$

Reduction and Proofs: $\mathrm{scd}(\langle s, t\rangle) \longrightarrow t$

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma \vdash s : \varphi\end{array} \qquad \begin{array}{c}\vdots \\ t \\ \vdots \\ \Gamma \vdash t : \psi\end{array}}{\Gamma \vdash \langle s, t\rangle : \varphi \wedge \psi}}{\Gamma \vdash t : \psi}$$

Reduction and Proofs: $\mathrm{scd}(\langle s, t \rangle) \longrightarrow t$

$$\begin{array}{c} \vdots \\ t \\ \vdots \\ \Gamma \vdash t : \psi \end{array}$$

Reduction and Proofs: $\text{cases}(\langle \text{lft}, s \rangle, t, r) \longrightarrow t\ s$

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma \vdash s : \varphi\end{array}}{\Gamma \vdash \langle \text{lft}, s \rangle : \varphi \vee \psi} \qquad \Gamma, x : \varphi \vdash t : \chi \qquad \Gamma, y : \psi \vdash r : \chi}{\Gamma \vdash \text{cases}(\langle \text{lft}, s \rangle, \lambda x\ .\ t, \lambda y\ .\ r) : \chi}$$

Reduction and Proofs: $\text{cases}(\langle \text{lft}, s\rangle, t, r) \longrightarrow t\ s$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma \vdash s : \varphi\end{array}}{\Gamma \vdash \langle \text{lft}, s\rangle : \varphi \vee \psi} \qquad \Gamma, x : \varphi \vdash t : \chi \qquad \Gamma, y : \psi \vdash r : \chi}{\Gamma \vdash (\lambda x \,.\, t)\ s : \chi}
$$

Reduction and Proofs: $\text{cases}(\langle \text{lft}, s \rangle, t, r) \longrightarrow t\ s$

$$\dfrac{\dfrac{\begin{matrix} \vdots \\ s \\ \vdots \\ \Gamma \vdash s : \psi \end{matrix}}{\Gamma \vdash \langle \text{rgt}, t \rangle : \varphi \vee \psi} \qquad \Gamma, x : \varphi \vdash t : \chi \qquad \Gamma, y : \psi \vdash r : \chi}{\Gamma \vdash \text{cases}(\langle \text{rgt}, t \rangle, \lambda x\ .\ t, \lambda y\ .\ r) : \chi}$$

Reduction and Proofs: $\text{cases}(\langle \text{lft}, s\rangle, t, r) \longrightarrow t\ s$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ s \\ \vdots \\ \Gamma \vdash s : \psi\end{matrix}}{\Gamma \vdash \langle \text{rgt}, t\rangle : \varphi \vee \psi} \qquad \Gamma, x : \varphi \vdash t : \chi \qquad \Gamma, y : \psi \vdash r : \chi}{\Gamma \vdash (\lambda y\ .\ r)\ s : \chi}$$

Reduction and Proofs: $(\lambda v\,.\,s)t \longrightarrow s[t/v]$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ s\\ \vdots\\ \Gamma, x\colon\varphi \vdash \psi\end{matrix}}{\Gamma \vdash \lambda x\,.\,s\colon \varphi\to\psi} \qquad \begin{matrix}\vdots\\ t\\ \vdots\\ \Gamma \vdash t\colon\varphi\end{matrix}}{\Gamma \vdash (\lambda x\,.\,s)t\colon \psi}$$

Reduction and Proofs: $(\lambda v\ .\ s)t \quad \longrightarrow \quad s[t/v]$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\\ s\\ \vdots\\ \Gamma, x\colon \varphi \vdash s\colon \psi\end{array}}{\Gamma \vdash \lambda x\ .\ s\colon \varphi \to \psi} \qquad \begin{array}{c}\vdots\\ t\\ \vdots\\ \Gamma \vdash t\colon \varphi\end{array}}{\Gamma \vdash s[t/x]\colon \psi}
$$

β-Reduction: $(\lambda v\ .\ s)t \longrightarrow s[t/v]$

$$(\lambda y\ .\ x + y^2 + y)\ 3$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$\begin{array}{ll} & (\lambda y \,.\, x + y^2 + y)\ 3 \\ \longrightarrow & x + 3^2 + 3 \end{array}$$

β-Reduction: $(\lambda v \, . \, s)t \quad \longrightarrow \quad s[t/v]$

$$
\begin{aligned}
& (\lambda y \, . \, x + y^2 + y) \; 3 \\
\longrightarrow \quad & x + 3^2 + 3 \\
\longrightarrow \quad & x + 9 + 3
\end{aligned}
$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$\begin{aligned} & (\lambda y \,.\, x + y^2 + y)\ 3 \\ \longrightarrow\quad & x + 3^2 + 3 \\ \longrightarrow\quad & x + 9 + 3 \\ \longrightarrow\quad & x + 12 \end{aligned}$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$(\lambda x \,.\, (\lambda y \,.\, x + y^2 + y)\ 3)\ 4$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

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\begin{array}{ll}
 & (\lambda x \,.\, (\lambda y \,.\, x + y^2 + y)\ 3)\ 4 \\
\longrightarrow & (\lambda x \,.\, x + 3^2 + 3)\ 4 \\
\longrightarrow & (\lambda x \,.\, x + 9 + 3)\ 4 \\
\longrightarrow & (\lambda x \,.\, x + 12)\ 4
\end{array}
$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$
\begin{array}{ll}
 & (\lambda x \,.\, (\lambda y \,.\, x + y^2 + y)\ 3)\ 4 \\
\longrightarrow & (\lambda x \,.\, x + 3^2 + 3)\ 4 \\
\longrightarrow & (\lambda x \,.\, x + 9 + 3)\ 4 \\
\longrightarrow & (\lambda x \,.\, x + 12)\ 4 \\
\longrightarrow & 4 + 12
\end{array}
$$

β-Reduction: $\quad (\lambda v\ .\ s)t \longrightarrow s[t/v]$

$$
\begin{aligned}
& (\lambda x\ .\ (\lambda y\ .\ x + y^2 + y)\ 3)\ 4 \\
\longrightarrow\ & (\lambda x\ .\ x + 3^2 + 3)\ 4 \\
\longrightarrow\ & (\lambda x\ .\ x + 9 + 3)\ 4 \\
\longrightarrow\ & (\lambda x\ .\ x + 12)\ 4 \\
\longrightarrow\ & 4 + 12 \\
\longrightarrow\ & 16
\end{aligned}
$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$(\lambda x \,.\, (\lambda y \,.\, x + y^2 + y)\ 3)\ 4$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$\begin{array}{ll} & (\lambda x \,.\, (\lambda y \,.\, x + y^2 + y)\ 3)\ 4 \\ \longrightarrow & (\lambda y \,.\, 4 + y^2 + y)\ 3 \end{array}$$

β-Reduction: $(\lambda v\ .\ s)t \longrightarrow s[t/v]$

$$
\begin{array}{ll}
 & (\lambda x\ .\ (\lambda y\ .\ x + y^2 + y)\ 3)\ 4 \\
\longrightarrow & (\lambda y\ .\ 4 + y^2 + y)\ 3 \\
\longrightarrow & 4 + 3^2 + 3
\end{array}
$$

β-Reduction: $(\lambda v\ .\ s)t \quad \longrightarrow \quad s[t/v]$

$$
\begin{array}{ll}
 & (\lambda x\ .\ (\lambda y\ .\ x + y^2 + y)\ 3)\ 4 \\
\longrightarrow & (\lambda y\ .\ 4 + y^2 + y)\ 3 \\
\longrightarrow & 4 + 3^2 + 3 \\
\longrightarrow & 4 + 9 + 3
\end{array}
$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$
\begin{array}{ll}
 & (\lambda x \,.\, (\lambda y \,.\, x + y^2 + y)\ 3)\ 4 \\
\longrightarrow & (\lambda y \,.\, 4 + y^2 + y)\ 3 \\
\longrightarrow & 4 + 3^2 + 3 \\
\longrightarrow & 4 + 9 + 3 \\
\longrightarrow & 16
\end{array}
$$

β-Reduction: $\quad (\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$(\lambda w \,.\, (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\ (\lambda z \,.\, z * w))\ 3\ 4$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$
\begin{array}{ll}
 & (\lambda w \,.\, (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\, (\lambda z \,.\, z * w))\ 3\ 4 \\
\longrightarrow & (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\, (\lambda z \,.\, z * 3)\ 4
\end{array}
$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$
\begin{array}{ll}
 & (\lambda w \,.\, (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\ (\lambda z \,.\, z * w))\ 3\ 4 \\
\longrightarrow & (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\ (\lambda z \,.\, z * 3)\ 4 \\
\longrightarrow & (\lambda y \,.\, (\lambda z \,.\, z * 3)\ (y^2 + y))\ 4
\end{array}
$$

β-Reduction: $(\lambda v\ .\ s)t \longrightarrow s[t/v]$

$$
\begin{array}{ll}
 & (\lambda w\ .\ (\lambda f\ .\ \lambda y\ .\ f(y^2 + y))\ (\lambda z\ .\ z * w))\ 3\ 4 \\
\longrightarrow & (\lambda f\ .\ \lambda y\ .\ f(y^2 + y))\ (\lambda z\ .\ z * 3)\ 4 \\
\longrightarrow & (\lambda y\ .\ (\lambda z\ .\ z * 3)\ (y^2 + y))\ 4 \\
\longrightarrow & (\lambda y\ .\ (y^2 + y) * 3)\ 4
\end{array}
$$

β-Reduction: $(\lambda v \, . \, s)t \longrightarrow s[t/v]$

$$
\begin{array}{ll}
& (\lambda w \, . \, (\lambda f \, . \, \lambda y \, . \, f(y^2 + y)) \, (\lambda z \, . \, z * w)) \; 3 \; 4 \\
\longrightarrow & (\lambda f \, . \, \lambda y \, . \, f(y^2 + y)) \, (\lambda z \, . \, z * 3) \; 4 \\
\longrightarrow & (\lambda y \, . \, (\lambda z \, . \, z * 3) \, (y^2 + y)) \; 4 \\
\longrightarrow & (\lambda y \, . \, (y^2 + y) * 3) \; 4 \\
\longrightarrow & (4^2 + 4) * 3 = (16 + 4) * 3 = 60
\end{array}
$$

β-Reduction: $(\lambda v \, . \, s)t \longrightarrow s[t/v]$

$$(\lambda y \, . \, (\lambda f \, . \, \lambda y \, . \, f(y^2 + y)) \, (\lambda z \, . \, z * y)) \; 3 \; 4$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$(\lambda y \,.\, (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\; (\lambda z \,.\, z * y))\; 3\; 4$$
$$\longrightarrow (\lambda y \,.\, \lambda y \,.\, (\lambda z \,.\, z * y)\; (y^2 + y))\; 3\; 4$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$
\begin{array}{ll}
& (\lambda y \,.\, (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\; (\lambda z \,.\, z * y))\; 3\; 4 \\
\longrightarrow & (\lambda y \,.\, \lambda y \,.\, (\lambda z \,.\, z * y)\; (y^2 + y))\; 3\; 4 \\
\longrightarrow & (\lambda y \,.\, \lambda y \,.\, (y^2 + y) * y)\; 3\; 4
\end{array}
$$

β-Reduction: $(\lambda v \,.\, s)t \longrightarrow s[t/v]$

$$
\begin{aligned}
& (\lambda y \,.\, (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\ (\lambda z \,.\, z * y))\ 3\ 4 \\
\longrightarrow\ & (\lambda y \,.\, \lambda y \,.\, (\lambda z \,.\, z * y)\ (y^2 + y))\ 3\ 4 \\
\longrightarrow\ & (\lambda y \,.\, \lambda y \,.\, (y^2 + y) * y)\ 3\ 4 \\
\longrightarrow\ & (\lambda y \,.\, (y^2 + y) * y)4
\end{aligned}
$$

β-Reduction: $(\lambda v\ .\ s)t \longrightarrow s[t/v]$

$$
\begin{aligned}
& (\lambda y\ .\ (\lambda f\ .\ \lambda y\ .\ f(y^2 + y))\ (\lambda z\ .\ z * y))\ 3\ 4 \\
\longrightarrow\ & (\lambda y\ .\ \lambda y\ .\ (\lambda z\ .\ z * y)\ (y^2 + y))\ 3\ 4 \\
\longrightarrow\ & (\lambda y\ .\ \lambda y\ .\ (y^2 + y) * y)\ 3\ 4 \\
\longrightarrow\ & (\lambda y\ .\ (y^2 + y) * y)4 \\
\longrightarrow\ & (4^2 + 4) * 4 = (16 + 4) * 4 = 80
\end{aligned}
$$

Rename bound variables to avoid capturing a free variable

$$(\lambda f\ .\ \lambda y\ .\ f(y^2 + y))\ (\lambda z\ .\ z * y)$$

Rename bound variables to avoid capturing a free variable

$$
\begin{array}{ll}
 & (\lambda f\ .\ \lambda y\ .\ f(y^2 + y))\ (\lambda z\ .\ z * y) \\
\longrightarrow & \lambda y_1\ .\ (\lambda z\ .\ z * y)\ (y_1^2 + y_1)
\end{array}
$$

Rename bound variables to avoid capturing a free variable

$$
\begin{array}{ll}
 & (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\,(\lambda z \,.\, z * y) \\
\longrightarrow & \lambda y_1 \,.\, (\lambda z \,.\, z * y)\,(y_1^2 + y_1) \\
\longrightarrow & \lambda y \,.\, \lambda y_1 \,.\, (y_1^2 + y_1) * y
\end{array}
$$

Free and bound variables

$$\begin{aligned} \mathrm{fv}(v) &= \{v\} \\ \mathrm{fv}(s\ t) &= \mathrm{fv}(s) \cup \mathrm{fv}(t) \\ \mathrm{fv}(\lambda v\ .\ s) &= \mathrm{fv}(s) \setminus \{v\} \end{aligned}$$

Free and bound variables

$$\begin{aligned} \mathsf{fv}(v) &= \{v\} \\ \mathsf{fv}(s\ t) &= \mathsf{fv}(s) \cup \mathsf{fv}(t) \\ \mathsf{fv}(\lambda v\ .\ s) &= \mathsf{fv}(s) \setminus \{v\} \end{aligned}$$

$$\begin{aligned} \mathsf{bv}(v) &= \emptyset \\ \mathsf{bv}(s\ t) &= \mathsf{bv}(s) \cup \mathsf{bv}(t) \\ \mathsf{bv}(\lambda v\ .\ s) &= \mathsf{fv}(s) \cup \{v\} \end{aligned}$$

Meaning of $s[t/v]$

If $\mathrm{fv}(t) \cap \mathrm{bv}(s) = \emptyset$, then plug in t in place of v in s

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If $\text{fv}(t) \cap \text{bv}(s) = \emptyset$, then plug in t in place of v in s

If $\text{fv}(t) \cap \text{bv}(s) \neq \emptyset$, then
change s:

1. replace all $v \in \text{fv}(t) \cap \text{bv}(s)$ with new variables throughout s
2. call this s'
3. return $s'[t/v]$

Rename bound variables to avoid capturing a free variable

$$
\begin{array}{ll}
 & (\lambda f \,.\, \lambda y \,.\, f(y^2 + y))\, (\lambda z \,.\, z * y) \\
\longrightarrow & \lambda y_1 \,.\, (\lambda z \,.\, z * y)\, (y_1^2 + y_1) \\
\longrightarrow & \lambda y \,.\, \lambda y_1 \,.\, (y_1^2 + y_1) * y
\end{array}
$$