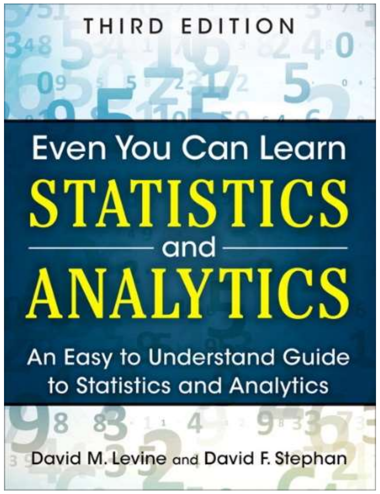
IMGD 2905

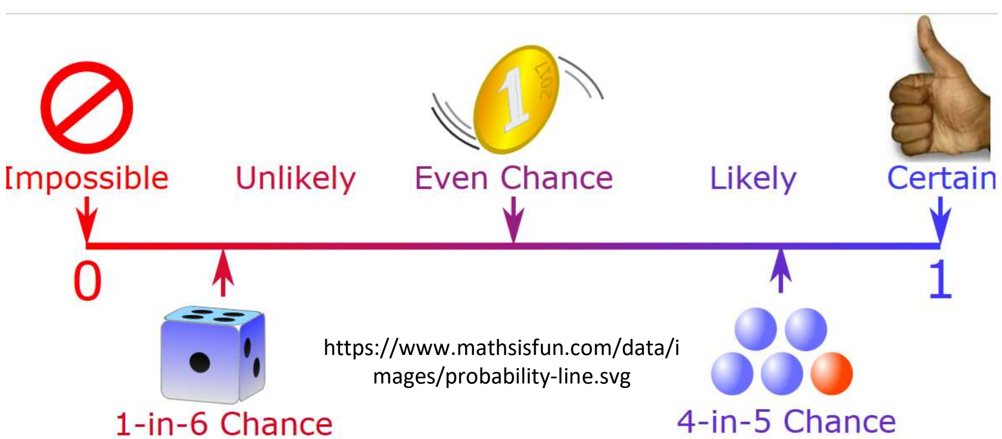
Probability

Chapters 4 & 5



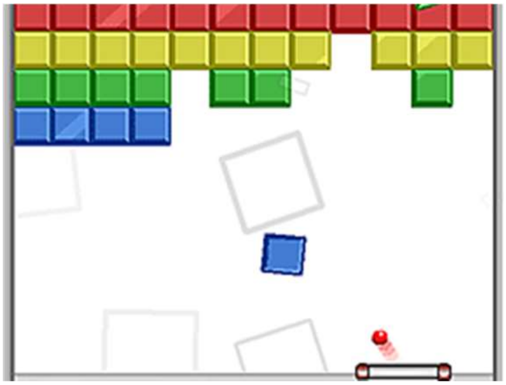
Overview

- Statistics important for game analysis
- Probability important for statistics
- So, understand some basic probability
- Also, probability useful for game development



https://www.mathsisfun.com/data/images/probability-line.svg

Breakout 5



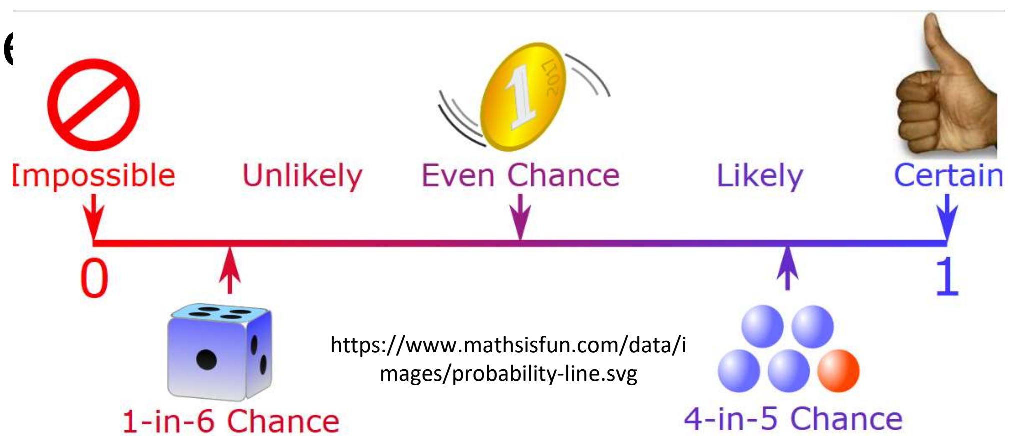
- Poll – group of 2 or group of 3?

- What are some examples of probabilities needed for game development?
- Provide a specific example
- Icebreaker, Groupwork, Questions

https://web.cs.wpi.edu/~imgd2905/d20/breakout/breakout-5.html

Overview

- Statistics important for game analysis
- Probability important for statistics
- So, understand some basic probability
- Also, probability useful for game



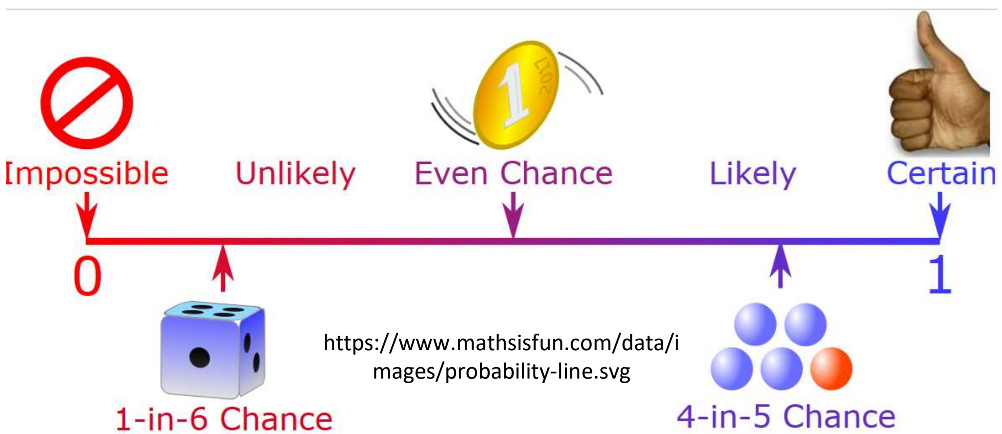
https://www.mathsisfun.com/data/images/probability-line.svg

- Probabilities for game development?
- Examples?

Overview

- Statistics important for game analysis
- Probability important for statistics
- So, understand some basic probability
- Also, probability useful for game development

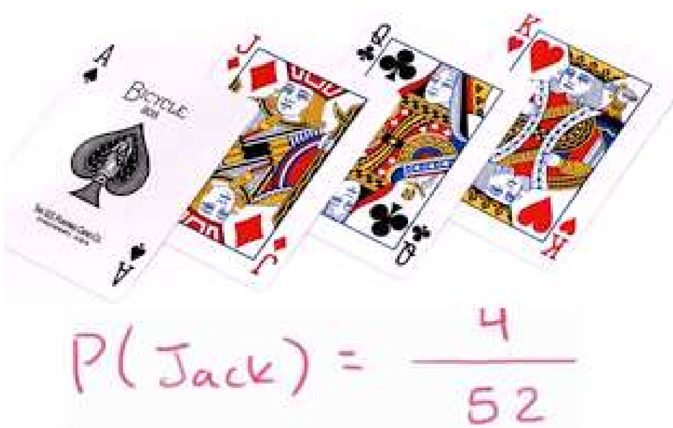
- Probability attack will succeed
- Probability loot from enemy contains rare item
- Probability enemy spawns at particular time
- Probability action (e.g., building a castle) takes particular amount of time
- Probability players at server



https://www.mathsisfun.com/data/images/probability-line.svg

Probability Introduction

- Probability – way of assigning numbers to outcomes to express likelihood of event
- Event – outcome of experiment or observation
  - Elementary – simplest type for given experiment. independent
  - Joint/Compound – more than one elementary



https://cdn.kastatic.org/googleusercontent/Z0TuLq2KolavsrfDXSbLqi0S-wnlCrC13cKGG68wK9ljrTiXzRqvfq7IpWNzcwgzlpEOI8YmMafp4K4zO0sanvXu

- Roll die (d6) and get 6
  - elementary event
- Roll die (d6) and get even number
  - compound event, consists of elementary events 2, 4, and 6
- Pick card from standard deck and get queen of spades
  - elementary event
- Pick card from standard deck and get face card
  - compound event
- Observe players logging in to MMO server and see if two people log in less than 15 minutes apart
  - compound event

We'll treat/compute probabilities of elementary versus compound separately

Outline

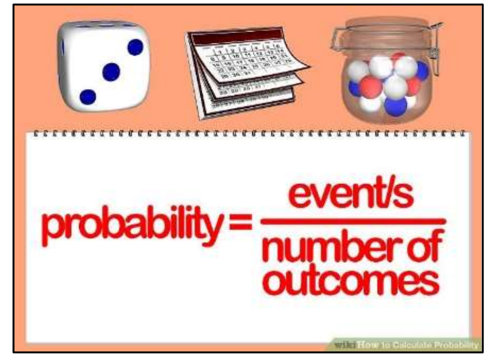
- Introduction (done)
- Probability (next)
- Probability Distributions

Probability – Definitions

- Exhaustive set of events – set of all possible outcomes of experiment/observation
- Mutually exclusive sets of events – elementary events that do not overlap
- Roll d6: Events: 1, 2
  - not exhaustive, mutually exclusive

- Roll d6: Events: 1, 2, 3, 4, 5, 6
  - exhaustive, mutually exclusive
- Roll d6: Events: get even number, get number divisible by 3, get a 1 or get a 5
  - exhaustive, but overlap
- Observe logins: time between arrivals <10 seconds, 10+ and <15 seconds inclusive, or 15+ seconds
  - exhaustive, mutually exclusive
- Observe logins: time between arrivals <10 seconds, 10+ and <15 seconds inclusive, or 10+ seconds
  - exhaustive, but overlap

Probability – Definition



https://goo.gl/iy3YGr

- Probability – likelihood of event to occur, ratio of favorable cases to all cases
- Set of rules that probabilities must follow
  - Probabilities must be between 0 and 1 (but often written/said as percent)
  - Probabilities of set of *exhaustive*, *mutually exclusive* events must add up to 1
- e.g., d6: events 1, 2, 3, 4, 5, 6. Probability of 1/6th to each, sum of P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1

  → legal set of probabilities
- e.g., d6: events 1, 2, 3, 4, 5, 6. Probability of ½ to roll 1, ½ to roll 2, and 0 to all the others sum of P(1) + … + P(6) = 0.5 + 0.5 + 0 … + 0 = 1

  → Also legal set of probabilities
  - Not how honest d6's behave in real life!

Q: how to assign probabilities?

How to Assign Probabilities?

http://static1.squarespace.com/static/5a14961cf14aa1f245bc3942/5a1c5e8d8165f542d6db3b0e/5acecc7f03ce64b9a46d99c6/1529981982981/Michael+Jordan+%2833%29.png?format=1500w

https://newvitruvian.com/images/marbles-clipart-bag-marble-4.png

Q: how to assign probabilities?

Assigning Probabilities

- Classical (by theory)
  - In many cases, exhaustive, mutually exclusive outcomes equally likely → assign each outcome probability of *1/n*
  - e.g., *d6*: 1/6, *Coin*: prob heads ½, tails ½, *Cards*: pick Ace 1/13
- Empirically (by observation)
  - Obtain data through measuring/observing
  - e.g., Watch how often people play FIFA 20 in FL222 versus some other game. Say, 30% FIFA. Assign that as probability
- Subjective (by hunch)
  - Based on expert opinion or other subjective method
  - e.g., eSports writer says probability Fnatic (European LoL team) will win World Championship is 25%

Rules About Probabilities (1 of 2)

- Complement: A an event. Event "Probability A does not occur" called *complement* of A, denoted A'

  P(A') = 1 - P(A)

  Q: why?

  – e.g., d6: P(6) = 1/6, complement is P(6') and probability of "not 6" is 1-1/6, or 5/6.

  – Note: Value often denoted p, complement is q

- Mutually exclusive: Have no simple outcomes in common – can't both occur in same experiment

  P(A or B) = P(A) + P(B)

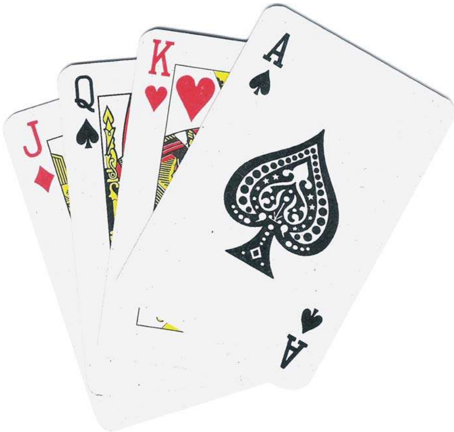
  – "Probability either occurs"

  – e.g., d6: P(3 or 6) = P(3) + P(6) = 1/6 + 1/6 = 2/6

Rules About Probabilities (2 of 2)

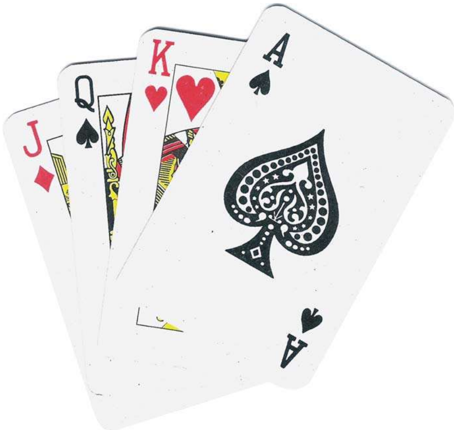
- Independent: Probability that one occurs doesn't affect probability that other occurs
  - e.g., 2d6: A= die 1 get 5, B= die 2 gets 6. Independent, since result of one roll doesn't affect roll of other
  - "Probability both occur" P(A and B) = P(A) x P(B)
  - e.g., 2d6: prob of "snake eyes" is P(1) x P(1) = 1/6 x 1/6 = 1/36
- Not independent: One occurs affects probability that other occurs
  - Probability both occur P(A and B) = P(A) x P(B | A)
    - Where P(B | A) means the prob B given A happened
  - e.g., LoL chance of getting most kills 20%. Chance of being support is 20%. You might think that:
    - P(kills) x P(support) = 0.2 x 0.2 = 0.04
  - But likely *not* independent. P(kills | support) < 20%. So, need non-independent formula
    - P(kills) * P(kills | support)

(Card example next slide)



Probability Example

- Probability drawing King?



Probability Example

- Probability drawing King?

  P(K) = ¼

- Draw, put back. Now?

Probability Example

- Probability drawing King?

  P(K) = ¼

- Draw, put back. Now?

  P(K) = ¼

- Probability *not* King?

Probability Example

- Probability drawing King?

  $P(K) = 1/4$
- Draw, put back. Now?

  $P(K) = 1/4$
- Probability *not* King?

  $P(K') = 1 - P(K) = 3/4$
- Draw, put back. 2 Kings?

Probability Example

- Draw. King or Queen?

- Probability drawing King?

  $P(K) = ¼$

- Draw, put back. Now?

  $P(K) = ¼$

- Probability *not* King?

  $P(K') = 1-P(K) = ¾$

- Draw, put back. Draw. 2 Kings?

  $P(K) \times P(K) = ¼ \times ¼ = 1/16$

Probability Example

- Draw. King or Queen?

  P(K or Q) = P(K) + P(Q)

  = ¼ + ¼ = ½

- Probability drawing King?

  P(K) = ¼

- Draw, put back. Now?

  P(K) = ¼

- Probability *not* King?

  P(K') = 1-P(K) = ¾

- Draw, put back. Draw. 2 Kings?

  P(K) x P(K) = ¼ x ¼ = 1/16

Probability Example

- Probability drawing King?

  P(K) = ¼

- Draw, put back. Now?

  P(K) = ¼

- Probability *not* King?

  P(K') = 1-P(K) = ¾

- Draw, put back. Draw. 2 Kings?

  P(K) x P(K) = ¼ x ¼ = 1/16

- Draw. King or Queen?

  P(K or Q) = P(K) + P(Q)

  = ¼ + ¼ = ½

- Draw, put back. Draw. Not King either card?

Probability Example

- Probability drawing King?

  $P(K) = \frac{1}{4}$
- Draw, put back. Now?

  $P(K) = \frac{1}{4}$
- Probability *not* King?

  $P(K') = 1 - P(K) = \frac{3}{4}$
- Draw, put back. Draw. 2 Kings?

  $P(K) \times P(K) = \frac{1}{4} \times \frac{1}{4} = 1/16$

- Draw. King or Queen?

  $P(K \text{ or } Q) = P(K) + P(Q)$

  $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$
- Draw, put back. Draw. Not King either card?

  $P(K') \times P(K') = \frac{3}{4} \times \frac{3}{4} = 9/16$
- Draw, *don't* put back. Draw. Not King either card?

Probability Example

- Probability drawing King?

  $P(K) = 1/4$
- Draw, put back. Now?

  $P(K) = 1/4$
- Probability *not* King?

  $P(K') = 1 - P(K) = 3/4$
- Draw, put back. 2 Kings?

  $P(K) \times P(K) = 1/4 \times 1/4 = 1/16$
- Draw. King or Queen?

  $P(K \text{ or } Q) = P(K) + P(Q)$

  $= 1/4 + 1/4 = 1/2$
- Draw, put back. Draw. Not King either card?

  $P(K') \times P(K') = 3/4 \times 3/4 = 9/16$
- Draw, *don't* put back. Draw. Not King either card?

  $P(K') \times P(K' \mid K') = 3/4 \times (1-1/3)$

  $= 3/4 \times 2/3$

  $= 6/12 = 1/2$
- Draw, don't put back. Draw. King 2nd card?

Probability Example

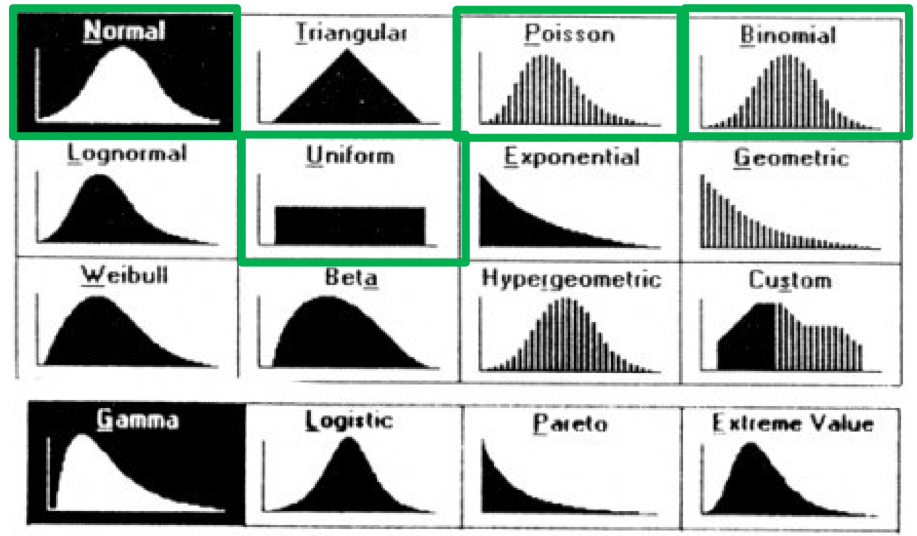
- Probability drawing King?
  $P(K) = 1/4$
- Draw, put back. Now?
  $P(K) = 1/4$
- Probability *not* King?
  $P(K') = 1 - P(K) = 3/4$
- Draw, put back. 2 Kings?
  $P(K) \times P(K) = 1/4 \times 1/4 = 1/16$
- Draw. King or Queen?
  $P(K \text{ or } Q) = P(K) + P(Q)$
  $= 1/4 + 1/4 = 1/2$
- Draw, put back. Draw. Not King either card?
  $P(K') \times P(K') = 3/4 \times 3/4 = 9/16$
- Draw, *don't* put back. Draw. Not King either card?
  $P(K') \times P(K' \mid K') = 3/4 \times (1-1/3)$
  $= 3/4 \times 2/3$
  $= 6/12 = 1/2$
- Draw, don't put back. Draw. King 2nd card?
  $P(K') \times P(K \mid K') = 3/4 \times 1/3 = 3/12 = 1/4$

Outline

- Intro (done)
- Probability (done)
- Probability Distributions (next)

Probability Distributions

- Probability distribution – values and likelihood of those values that random variable can take
- Why? If can model mathematically, can use to predict occurrences
  - e.g., probability slot machine pays out on given day
  - e.g., probability game server hosts player today
  - e.g., probability certain game mode is chosen by player
  - Also, some statistical techniques for some distributions only



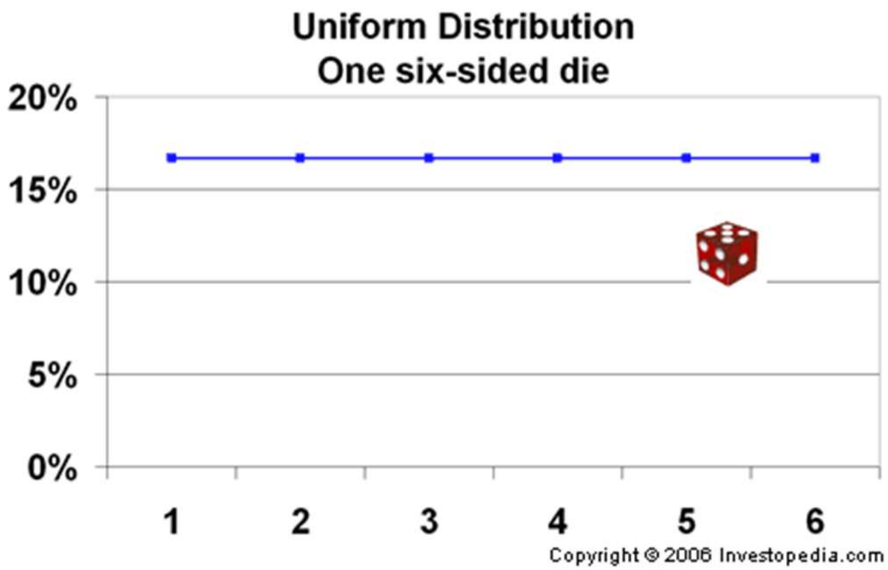
https://goo.gl/jqomFI

Types discussed:

Uniform (discrete)
Binomial (discrete)
Poisson (discrete)
Normal (continuous)

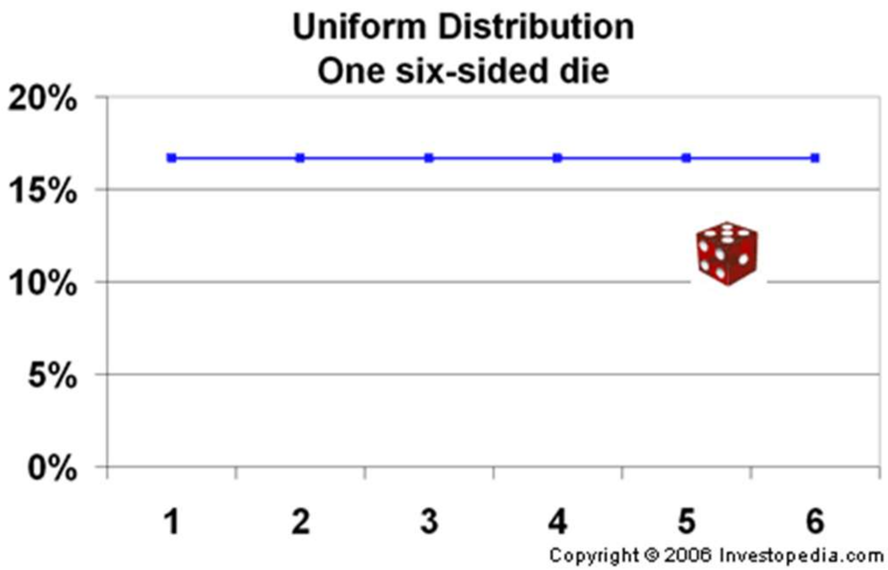
Remember empirical rule?
What distribution did it apply to?

Uniform Distribution



- "So what?"
- Can use known formulas

Uniform Distribution



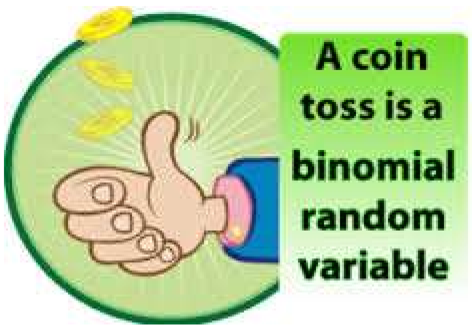
- "So what?"
- Can use known formulas

Mean	$\frac{a+b}{2}$
Median	$\frac{a+b}{2}$
Mode	N/A
Variance	$\frac{(b-a+1)^2-1}{12}$

Mean = (1 + 6) / 2 = 3.5
Variance = $((6 - 1 + 1)^2 - 1)/12$
= 2.9
Std Dev = sqrt(Variance) = 1.7

Note – mean is also the expected value
(if you did a lot of trials, would be average result)

Binomial Distribution Example (1 of 3)



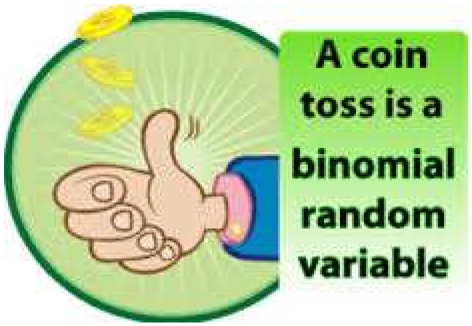
How to assign probabilities?

- Suppose toss 3 coins
- Random variable

  X = number of heads
- Want to know probability of *exactly* 2 heads

  $P(X=2) = ?$

Binomial Distribution Example (1 of 3)



- Suppose toss 3 coins
- Random variable

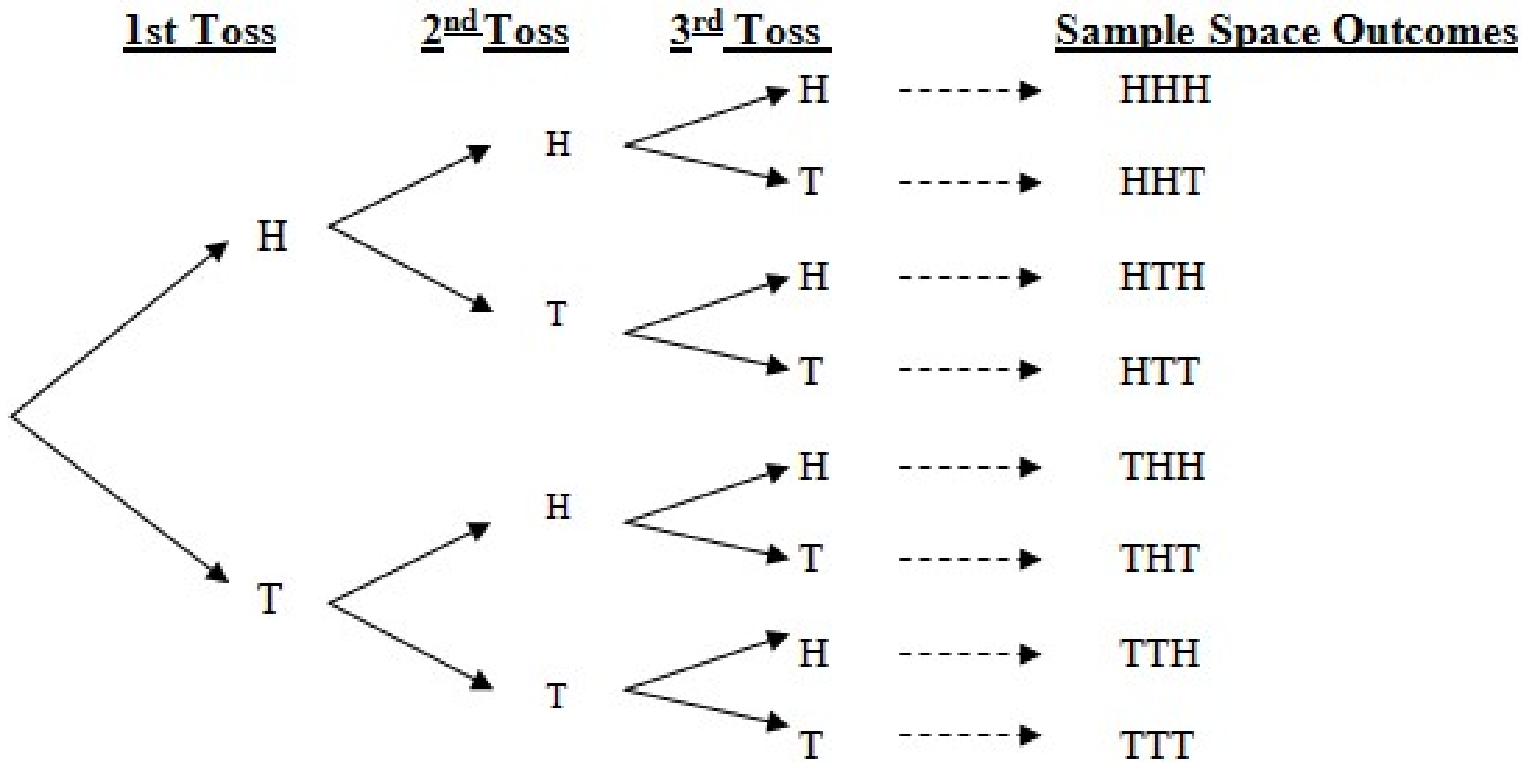
  X = number of heads
- Want to know probability of *exactly* 2 heads

  P(X=2) = ?

How to assign probabilities?

- Could *measure* (empirical)
  - *Q: how?*
- *Could use "hunch"* (subjective)
  - *Q: what do you think?*
- *Could use theory* (classical)
  - *Math using our probability* rules (not shown)
  - Enumerate (next)

Binomial Distribution Example (2 of 3)

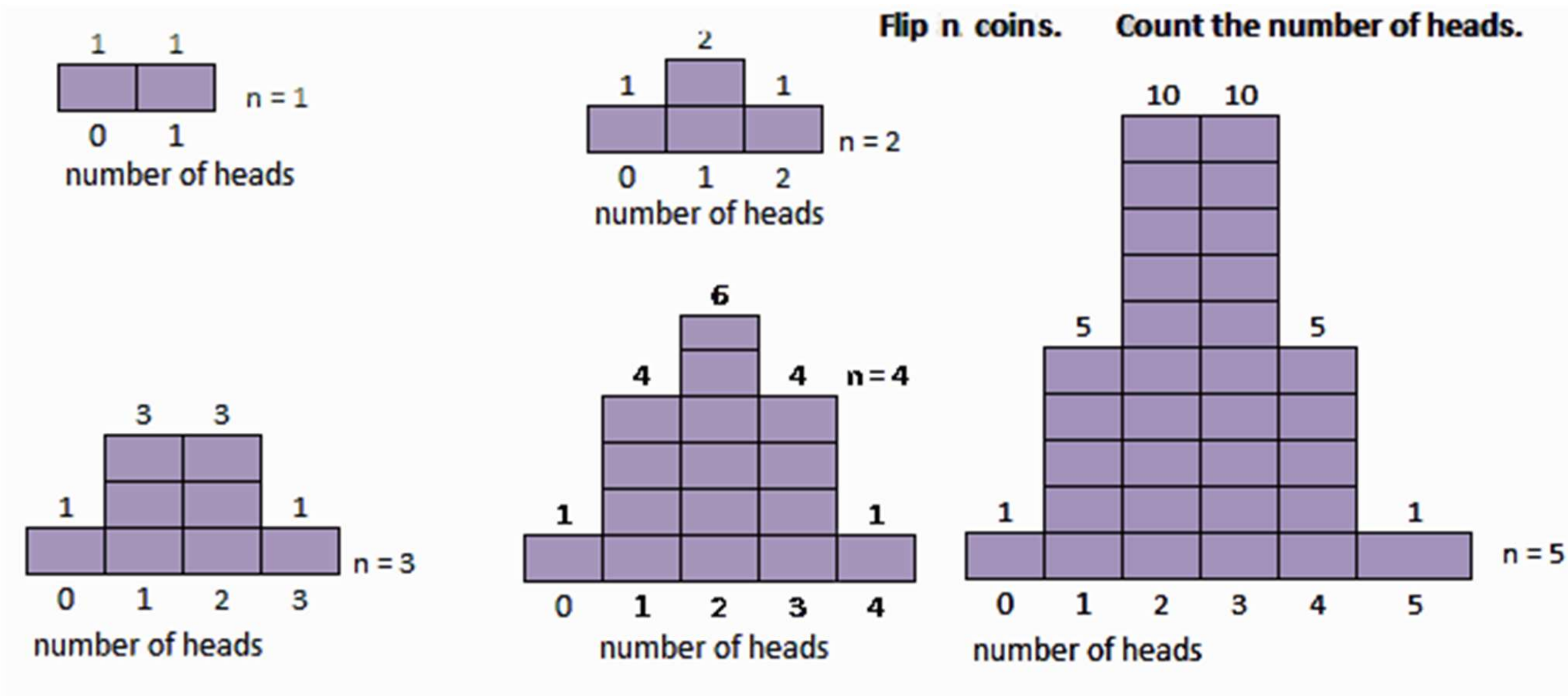


http://web.mnstate.edu/peil/MDEV102/U3/S25/Cartesian3.PNG

All equally likely (p is 1/8 for each)
→ P(HHT) + P(HTH) + P(THH) = 3/8

Can draw histogram of number of heads

Binomial Distribution Example (3 of 3)



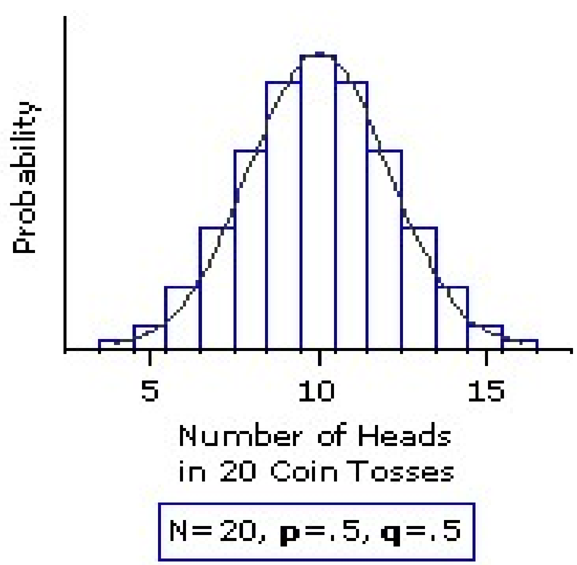
http://www.mathnstuff.com/math/spoken/here/2class/90/binom2.gif

These are all binomial distributions

Note, again expected value -
average amount you'd get if you did many trials

Binomial Distribution (1 of 2)

- In general, any number of trials (n) & any probability of successful outcome (p) (e.g., heads)



http://www.vassarstats.net/textbook/f0603.gif

- Characteristics of experiment that gives random number with binomial distribution:
  - Experiment consists of n identical trials.
  - Trials are independent
  - Each trial results in only two possible outcomes, S or F
  - Probability of S each trial is same, denoted p
  - Random variable of interest (X) is number of S's in n trials

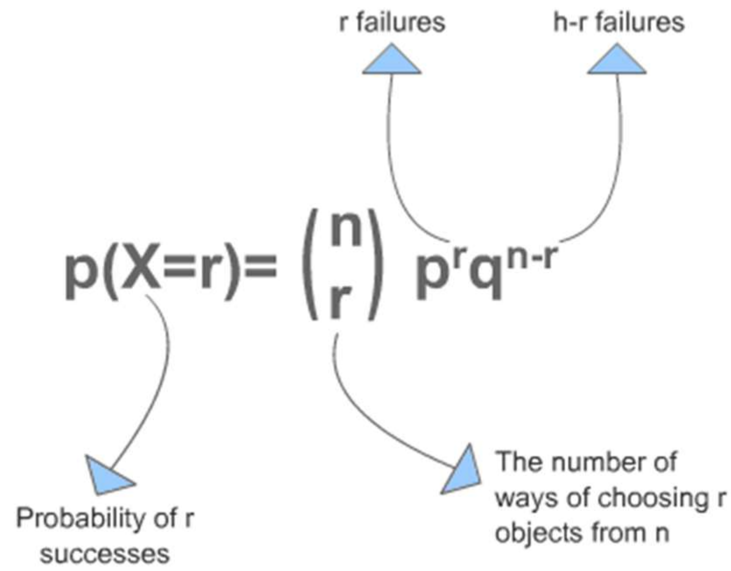
Binomial Distribution (2 of 2)

- "So what?"
- Can use known formulas

$$MEAN: \quad \mu = np$$

$$Variance: \quad \sigma^2 = npq$$

$$SD: \quad \sigma = \sqrt{npq}$$



http://www.s-cool.co.uk/gifs/a-mat-sdisc-dia08.gif

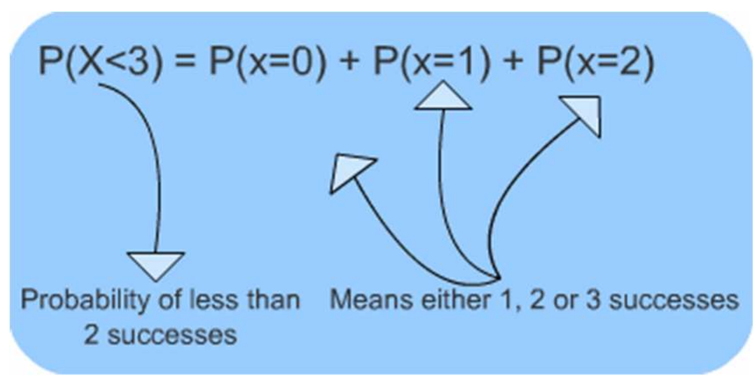
Excel: `binom.dist()`

`binom.dist(x,trials,prob,cumulative)`

– 2 heads, 3 flips, coin, discrete

`=binom.dist(2,3,0.5,FALSE)`

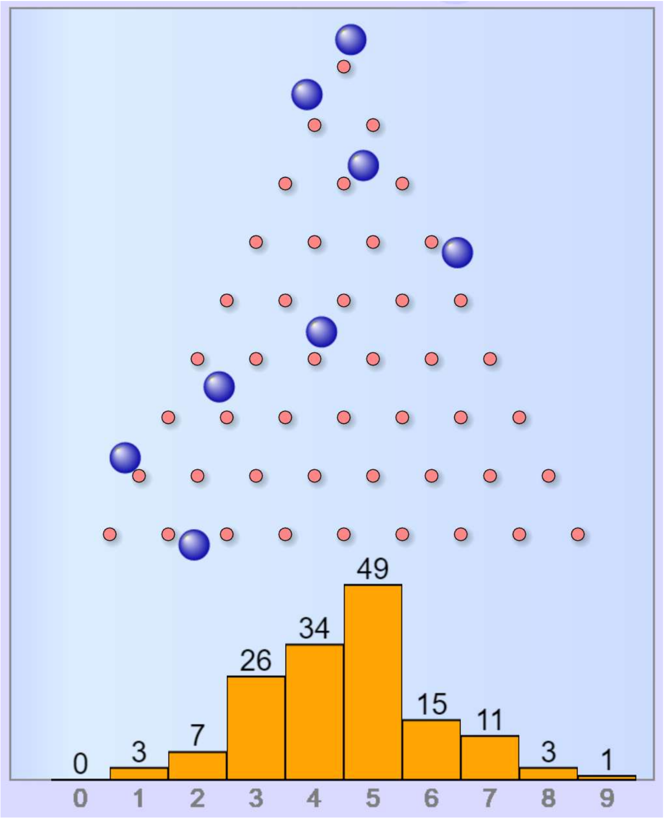
`=0.375` (i.e., 3/8)



http://www.s-cool.co.uk/gifs/a-mat-sdisc-dia12.gif

Binomial Distribution Example

- Each row is like a coin flip
  - right = “heads”
  - left = “tails”
- Bottom axis is number of heads
- Can compute P(X) by:
  - bin(X) / sum(bin(0) + bin(1) + …)



https://www.mathsisfun.com/data/quincunx.html

Poisson Distribution

- Distribution of probability of events occurring in certain interval (broken into units)
  - Interval can be time, area, volume, distance
  - e.g., number of players arriving at server lobby in 5-minute period between noon-1pm
- Requires
  1. Probability of event same for all time units
  2. Number of events in one time unit independent of number of events in any other time unit
  3. Events occur singly (not simultaneously). In other words, as time unit gets smaller, probability of two events occurring approaches 0

Poisson Distributions?

Not Poisson

- Number of people arriving at restaurant during dinner hour
  - People frequently arrive in groups
- Number of students register for course in BannerWeb per hour on first day of registration
  - Prob not equal – most register in first few hours
  - Not independent – if too many register early, system crashes

Could Be Poisson

- Number of groups arriving at restaurant during dinner hour
- Number of logins to MMO during prime time
- Number of defects (bugs) per 100 lines of code
- People arriving at cash register (if they shop individually)

Phrase people use is random arrivals

Poisson Distribution

- Distribution of probability of events occurring in certain interval

$$P(X=x)=e^{-\lambda}\frac{\lambda^{x}}{x!}$$

- $X =$ a Poisson random variable
- $x =$ number of events whose probability you are calculating
- $\lambda =$ the Greek letter "lambda," which represents the average number of events that occur per time interval
- $e =$ a constant that's equal to approximately 2.71828

http://www.dummies.com/education/math/business-statistics/how-to-compute-poisson-probabilities/

Poisson Distribution Example

1. Number of games student plays per day averages 1 per day
2. Number of games played per day independent of all other days
3. Can only play one game at a time

- What's probability of playing 2 games tomorrow?
- In this case, the value of $\lambda = 1$, want $P(X=2)$

$$P\left(X=2\right)=e^{-1}\frac{1^2}{2!}=0.1839$$

Poisson Distribution

- "So what?" → Known formulas

$$P(X = x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

- Mean = λ
- Variance = λ
- Std Dev = sqrt (λ)

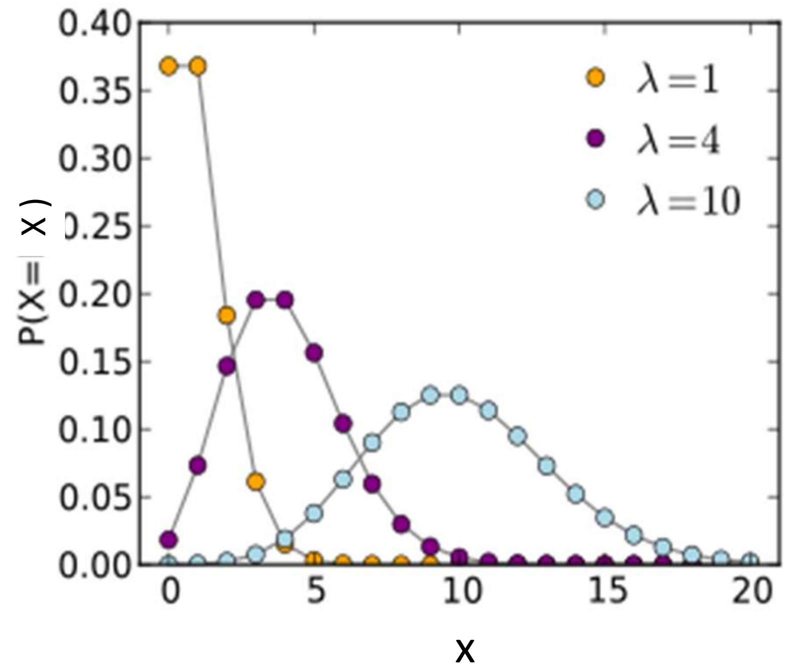
Excel: `poisson.dist()`

`poisson.dist(x,mean,cumulative)`

mean 1 game per day, chance for 2?

`= poisson.dist(2,1,false)`

`= 0.18394`



e.g., May want to know most likelihood of 1.5x average people arriving at server

Expected Value

- Expected value of discrete random variable is value you'd *expect* after many experimental trials. i.e., mean value of population

Value:	x_1	x_2	x_3	…	x_n
Probability:	$P(x_1)$	$P(x_2)$	$P(x_3)$	…	$P(x_n)$

- Compute by multiplying each by probability and summing

$$\boxed{\begin{aligned}\mu_x = E(X) &= x_1P(x_1) + x_2P(x_2) + \ldots + x_nP(x_n) \\ &= \Sigma\, x_iP(x_i)\end{aligned}}$$

Expected Value Example – Gambling Game

- Pay $3 to enter
- Roll 1d6 → 6? Get $7 1-5? Get $1
- What is expected payoff?

Outcome	Payoff	P(x)	xP(x)
1-5	$1		
6	$7		

Expected Value Example – Gambling Game

- Pay \$3 to enter
- Roll 1d6 → 6? Get \$7 1-5? Get \$1
- What is expected payoff?

Outcome	Payoff	P(x)	xP(x)
1-5	\$1	5/6	\$5/6
6	\$7	1/6	\$7/6

E(x) =

Expected Value Example – Gambling Game

- Pay $3 to enter
- Roll 1d6 → 6? Get $7 1-5? Get $1
- What is expected payoff? Expected net?

Outcome	Payoff	P(x)	xP(x)
1-5	$1	5/6	$5/6
6	$7	1/6	$7/6

E(x) = $5/6 + $7/6 = $12/6 = $2

Expected Value Example – Gambling Game

- Pay \$3 to enter
- Roll 1d6 → 6? Get \$7   1-5? Get \$1
- What is expected payoff? Expected net?

Outcome	Payoff	P(x)	xP(x)
1-5	\$1	5/6	\$5/6
6	\$7	1/6	\$7/6

E(x) = \$5/6 + \$7/6 = \$12/6 = \$2

E(net) = E(x) - \$3 = \$2 - \$3 = \$-1