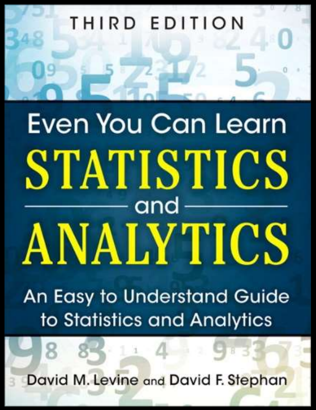
IMGD 2905

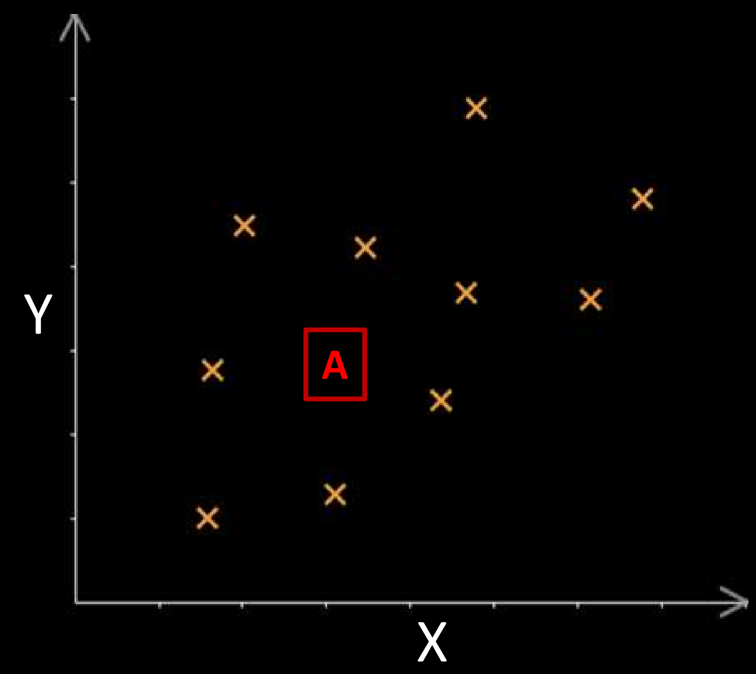
Simple Linear Regression

Chapter 10



Motivation GA

- Have data (sample, **x**'s)
- Want to know likely value of next observation (Y)
  - E.g., *playtime*
- A Q: Given previous Y's, what is likely next Y?



Motivation

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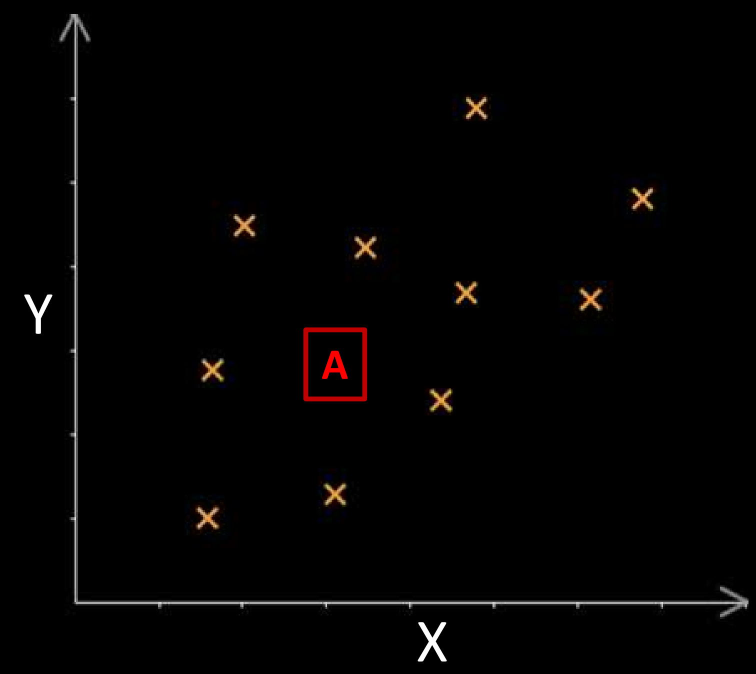
  - E.g., *playtime skins owned*
- A – reasonable to compute mean (with confidence interval)



Motivation

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- Want to know likely value of next observation (Y)
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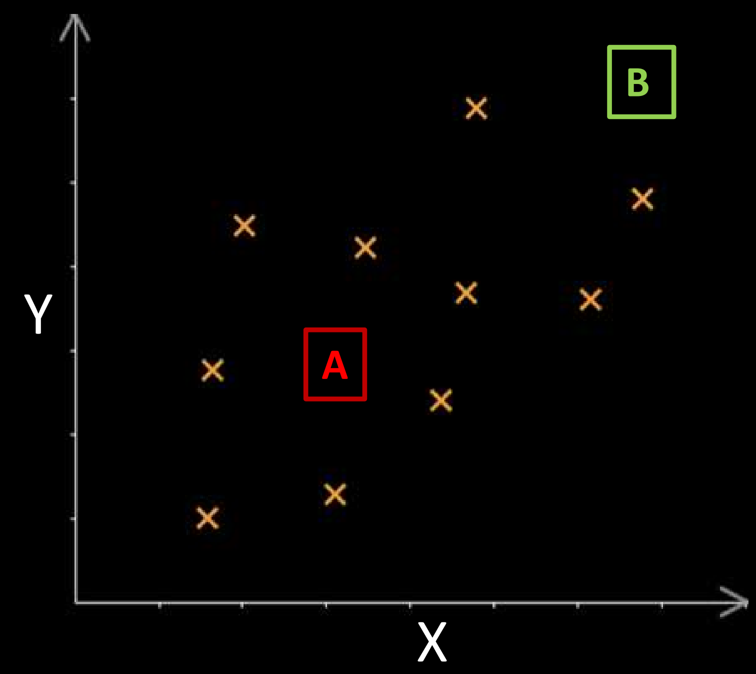
- A – reasonable to compute mean (with confidence interval)
- B – could do same?



Motivation

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- Want to know likely value of next observation (Y)
  - E.g., *playtime skins owned*
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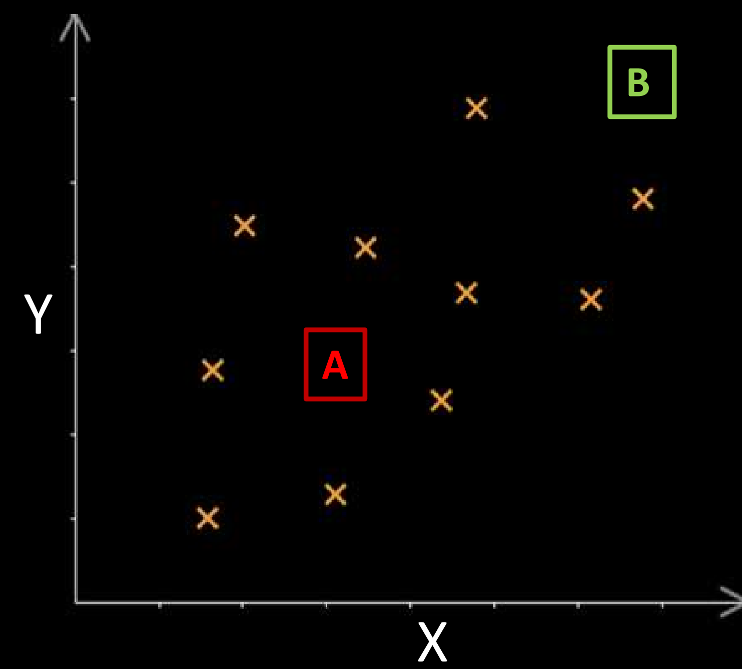
- B – could do same, but there appears to be relationship between X and Y!

→ *Predict* B

e.g., "trendline" (regression)



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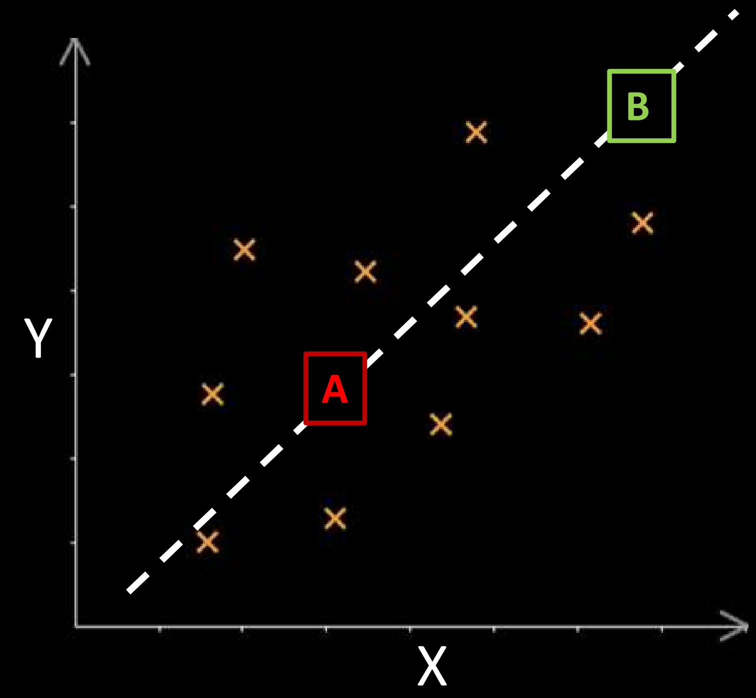
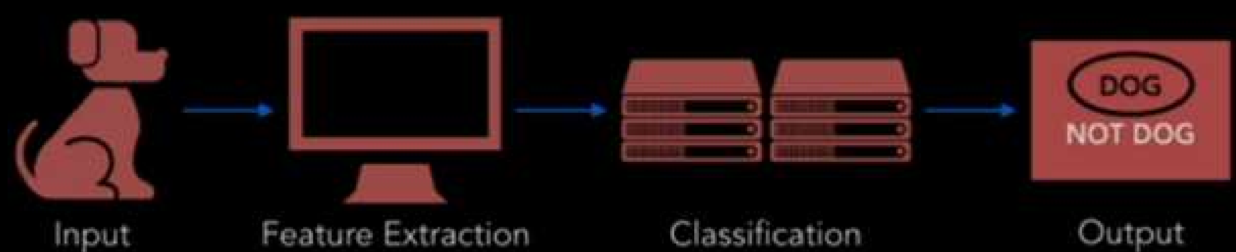


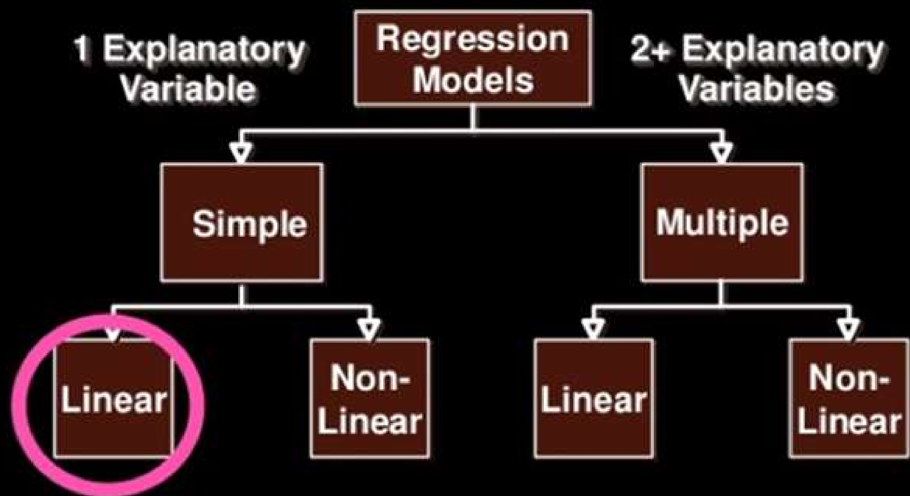
Overview

Broadly, two types of prediction techniques:

1. Regression – mathematical equation to model, use model for predictions
   – We'll discuss simple linear regression
2. Machine learning – branch of AI, use computer algorithms to determine relationships (predictions)
   – CS 4342 Machine Learning



Types of Regression Models



- Explanatory variable *explains* dependent variable
  - Variable X (e.g., skill level) explains Y (e.g., KDA)
  - Can have 1 (simple) or 2+ (multiple)
- Linear if coefficients added, else Non-linear

Outline GA

- Introduction (done)
- Simple Linear Regression (next)
  - Linear relationship
  - Residual analysis
  - Fitting parameters
- Measures of Variation
- Misc

Simple Linear Regression

- Goal – find a linear (line) relationship between two values
  - E.g., *KDA* and *skill*, *time* and *car speed*
- First, make sure relationship is linear! How?

Simple Linear Regression

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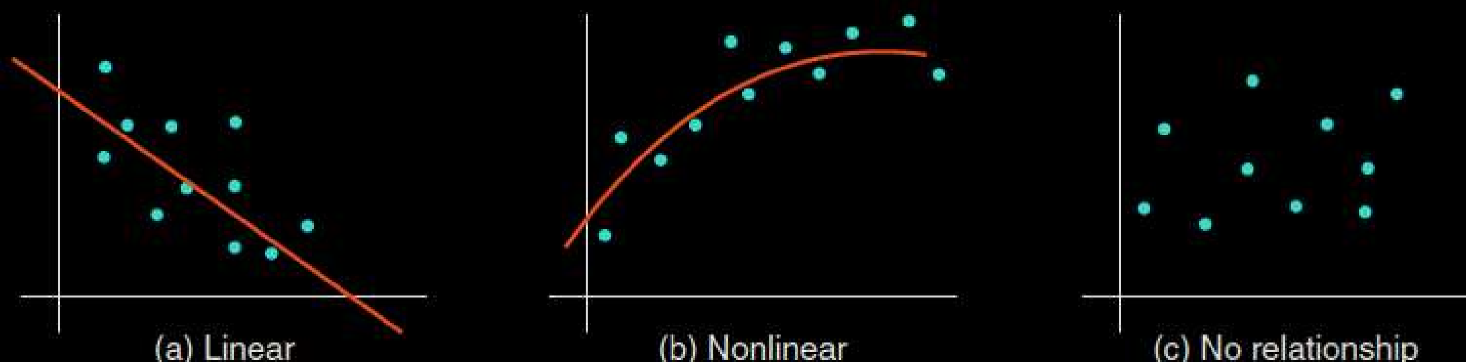
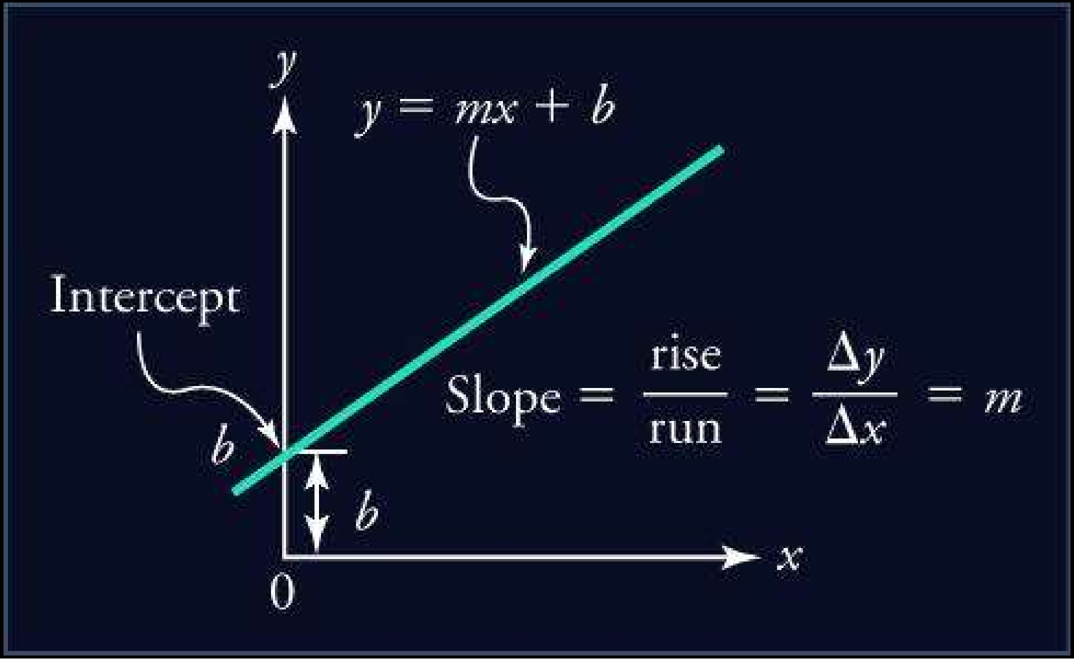
→ Scatterplot

(c) no clear relationship
(b) not a linear relationship
(a) linear relationship – proceed with linear regression



Linear Relationship

- From algebra: line in form $Y = mX + b$
  - m is slope, b is y-intercept
- Slope (m) is amount *Y* increases when *X* increases by 1 unit
- Intercept (b) is where line crosses y-axis, or y-value when *x* = 0



Simple Linear Regression Example

- Size of house related to its market value
  - *X* = square footage
  - *Y* = market value ($)
- Scatter plot (42 homes) indicates linear trend

	A	B	C
1	Home Market Value		
2			
3	House Age	Square Feet	Market Value
4	33	1,812	$90,000.00
5	32	1,914	$104,400.00
6	32	1,842	$93,300.00
7	33	1,812	$91,000.00
8	32	1,836	$101,900.00
9	33	2,028	$108,500.00
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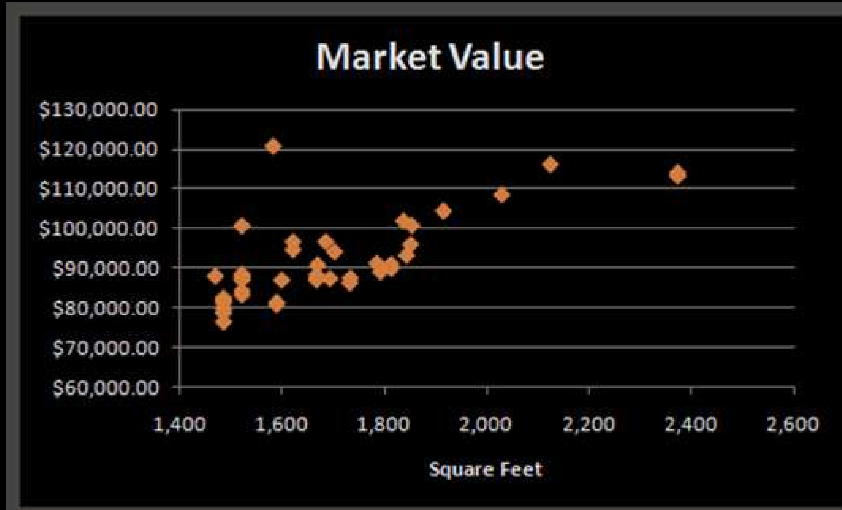
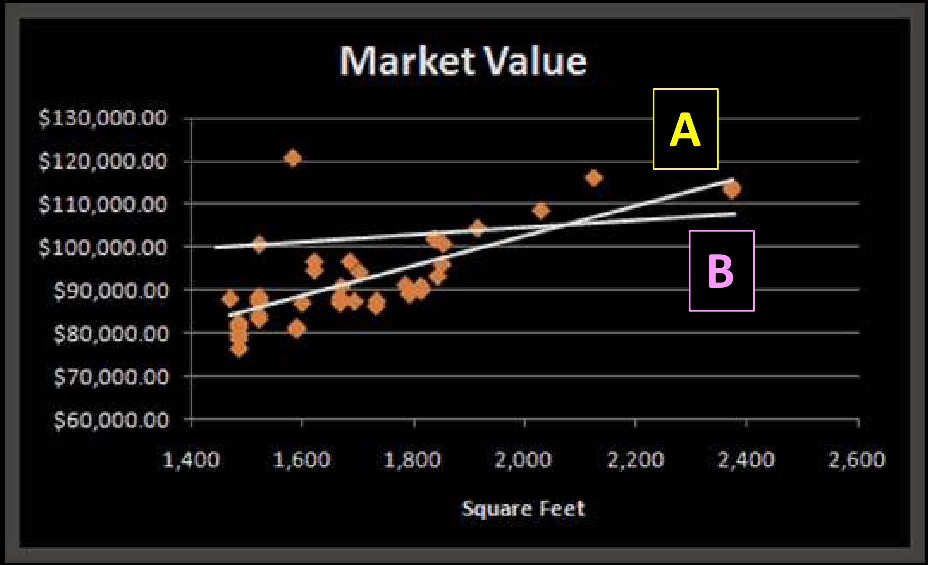


Simple Linear Regression Example

- Two possible lines shown below (A and B)
- Want to determine best regression line
- Line A looks a better fit to data
  - But how to know?

$$Y = mX + b$$

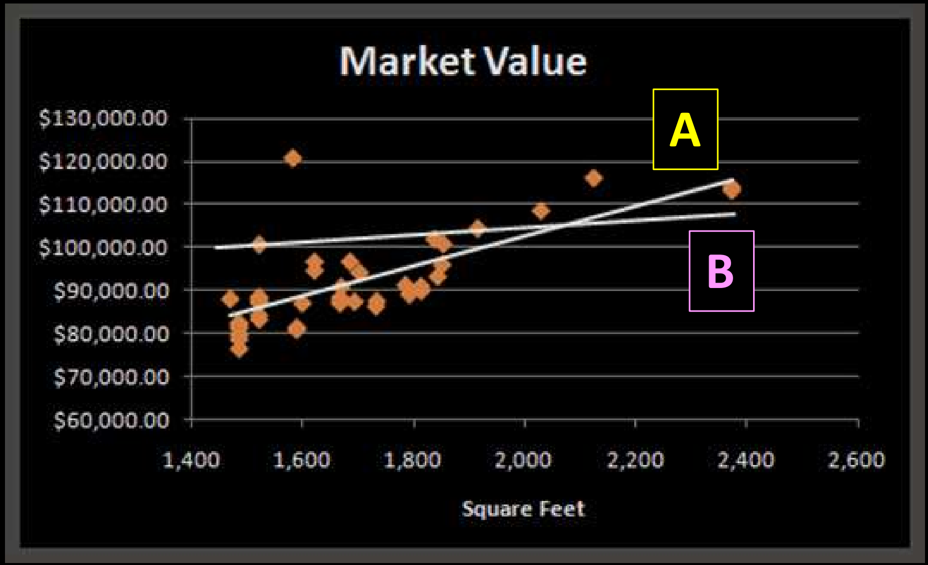


Simple Linear Regression Example

- Two possible lines shown below (A and B)
- Want to determine best regression line
- Line A looks a better fit to data
  - But how to know?

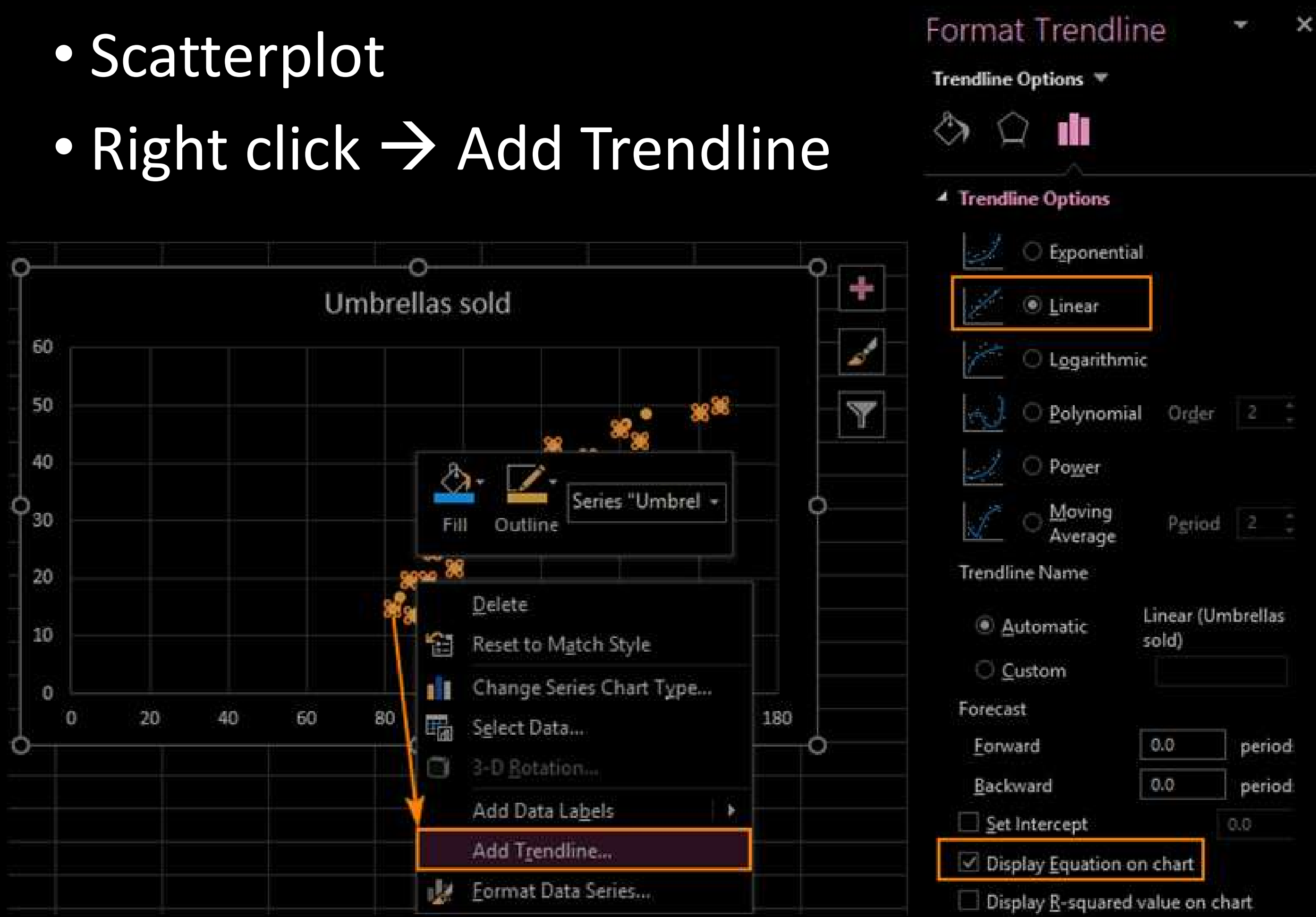
$Y = mX + b$

Line that gives best fit to data is one that minimizes prediction error
→Least squares line (more later)



Simple Linear Regression Example Chart

- Scatterplot
- Right click → Add Trendline



Simple Linear Regression Example
Formulas

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`=SLOPE(C4:C45,B4:B45)`

- Slope → 35.036

`=INTERCEPT(C4:C45,B4:B45)`

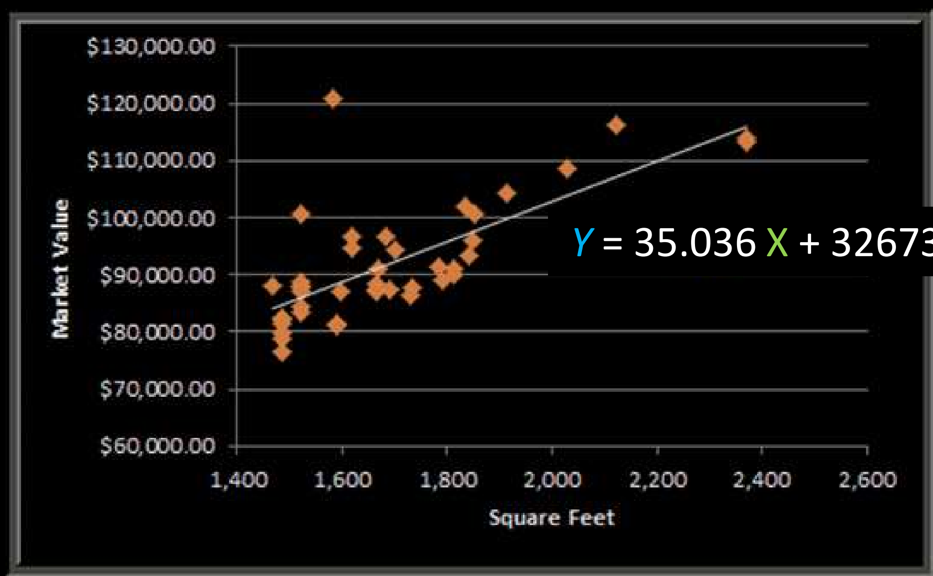
- Intercept → 32,673
- Estimate Y when X = 1800 square feet?

Y = 35.036 x (1800) + 32,673 = $95,737.80

Simple Linear Regression Example

Market value = 32673 + 35.036 *x* (square feet)

- Predicts market value better than just average



But before use, examine residuals

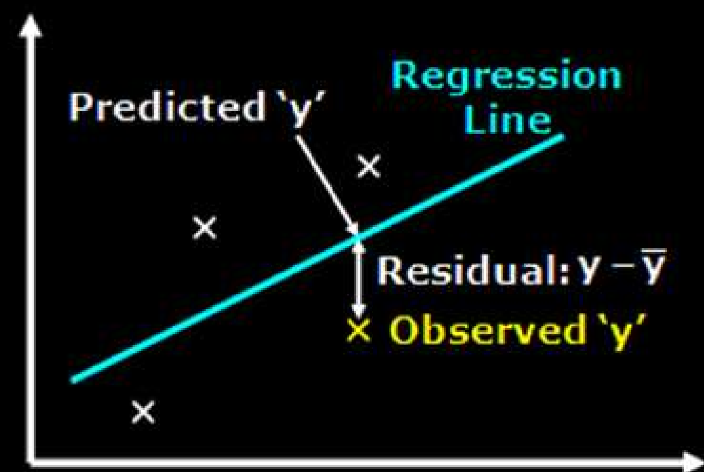
Outline

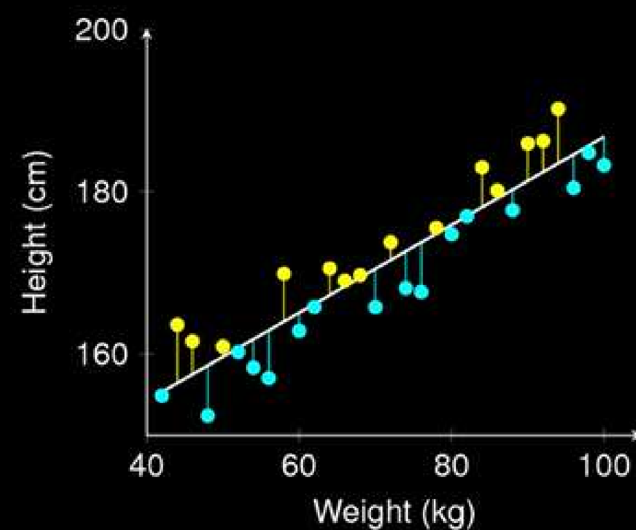
- Introduction (done)
- Simple Linear Regression
  - Linear relationship (done)
  - Residual analysis (next)
  - Fitting parameters
- Measures of Variation
- Misc

Residual Analysis

- Before predicting, confirm linear regression assumptions hold
    1. Variation around line is normally distributed
    2. Variation equal for all *X*
    3. Variation independent for all *X*
- How? Compute residuals (error in prediction)

→ Chart

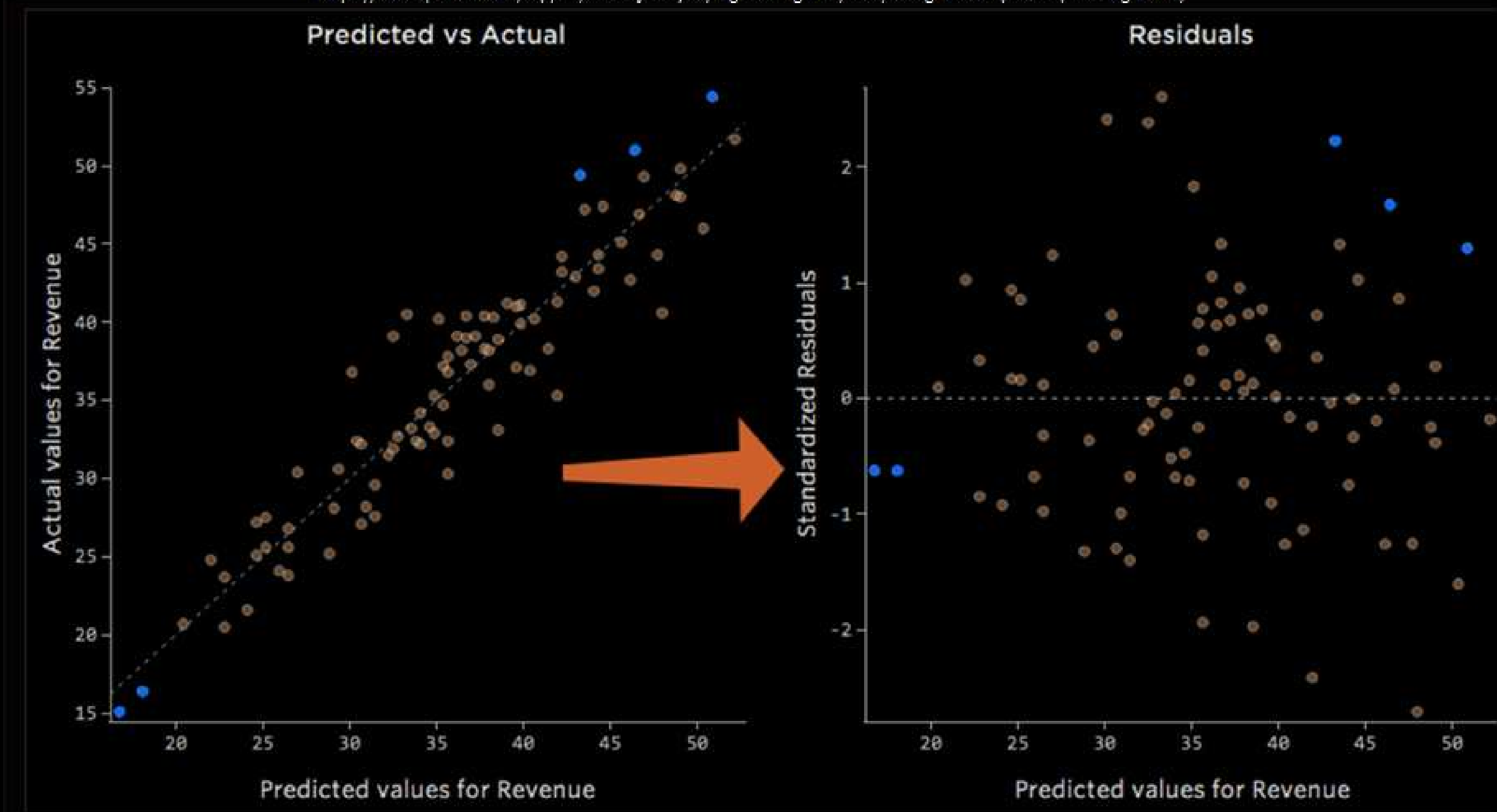




Residual Analysis

Variation around line normally distributed?
Variation equal for all X?
Variation independent for all X?

https://www.qualtrics.com/support/stats-iq/analyses/regression-guides/interpreting-residual-plots-improve-regression/

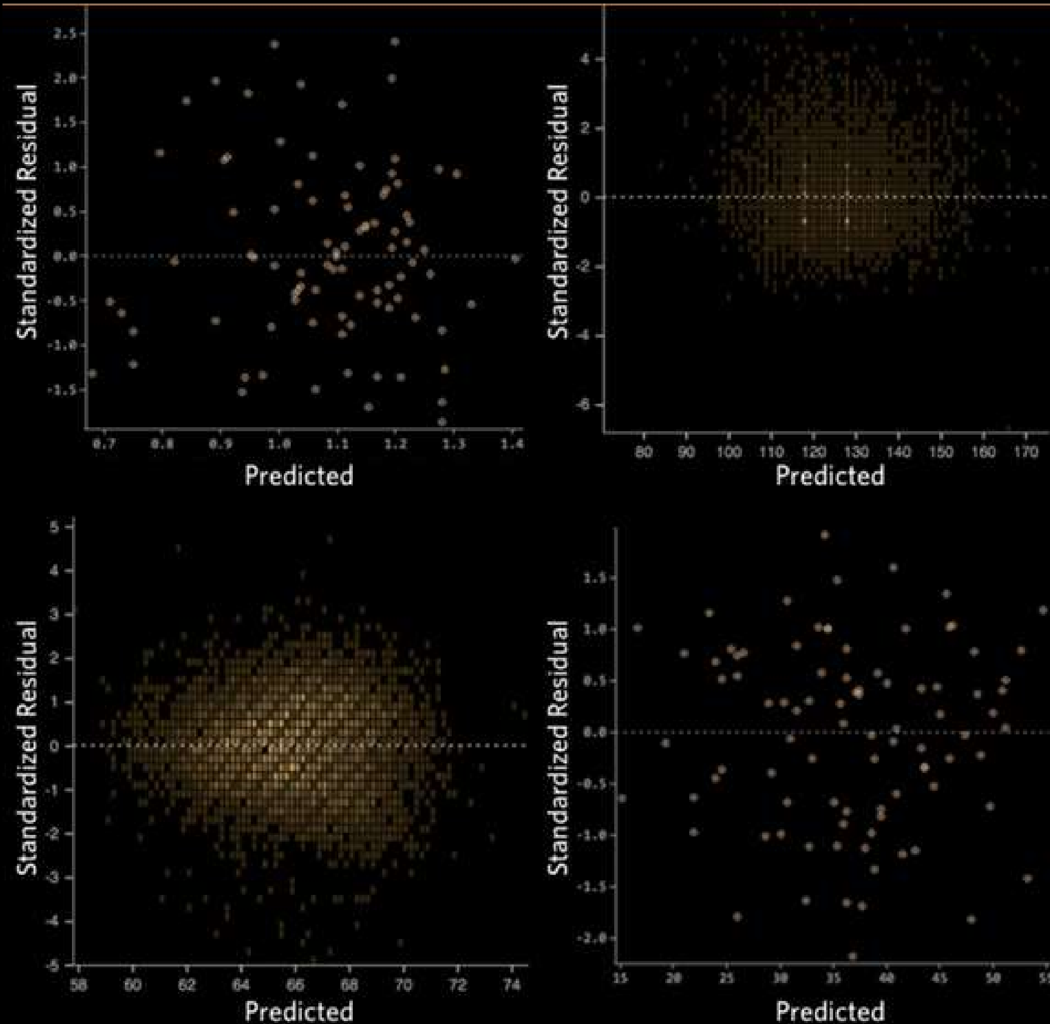


Note that we've colored in a few dots in orange so you can get the sense of how this transformation works.

Residual Analysis – Good

Clustered towards middle

Symmetrically distributed



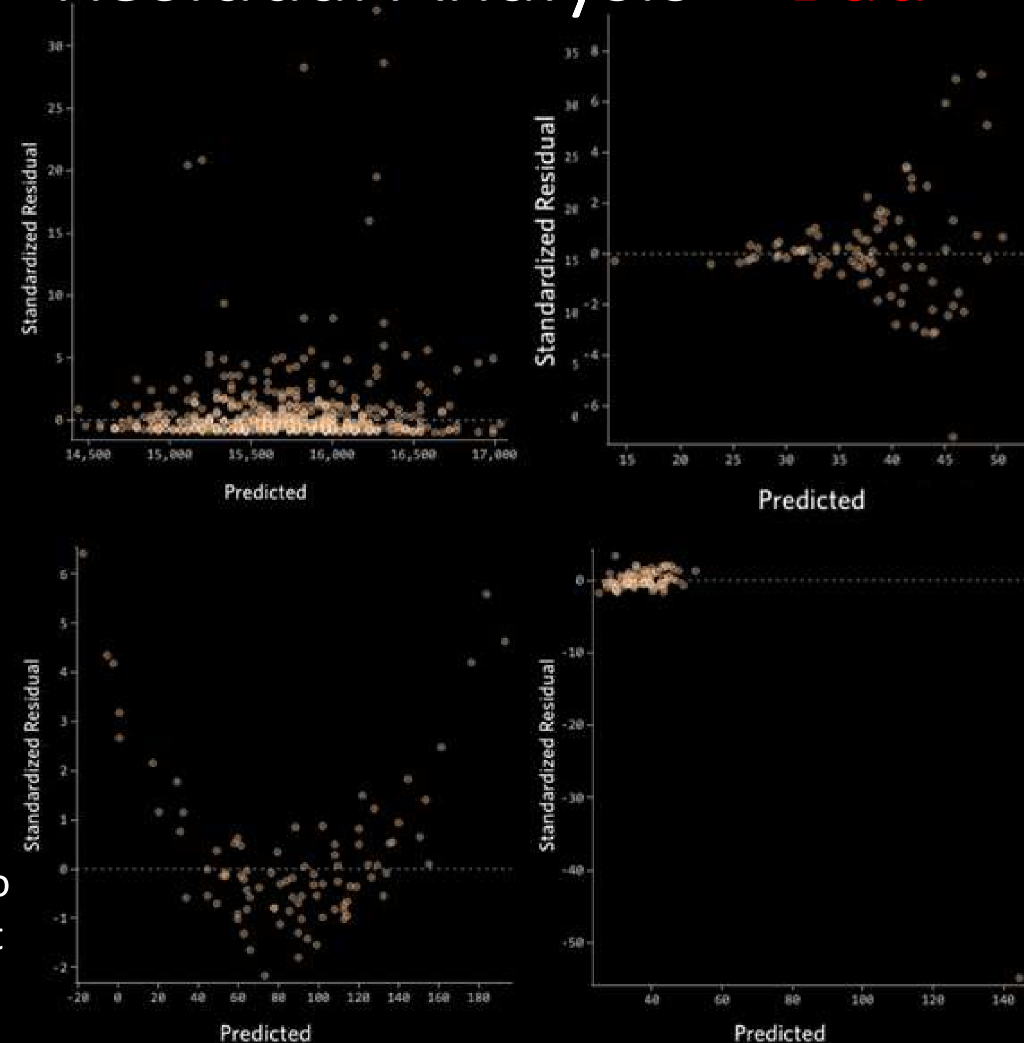
No clear pattern

Residual Analysis – Bad

Clear shape

Outliers

Patterns



Note: could do normality test (QQ plot)

Residual Analysis – Summary

- Regression assumptions:
  1. Normality of variation around regression
  2. Equal variation for all y values
  3. Independence of variation

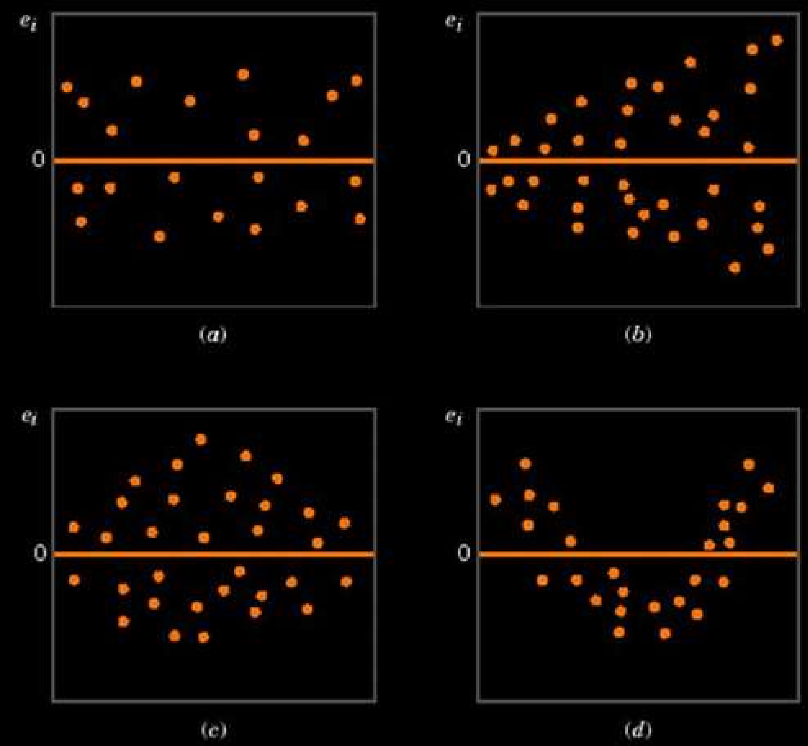
  ____

  (a) ok

  (b) funnel

  (c) double bow

  (d) nonlinear



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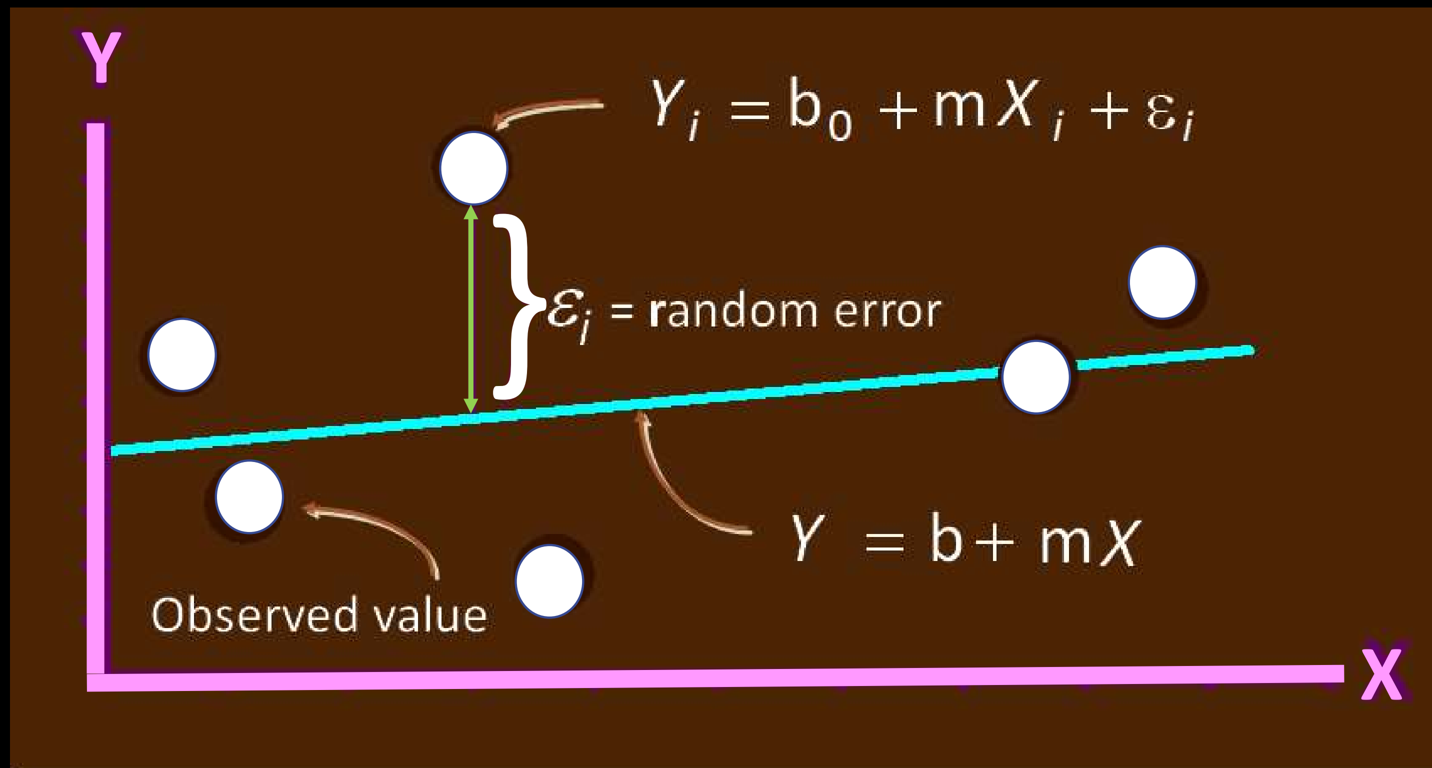
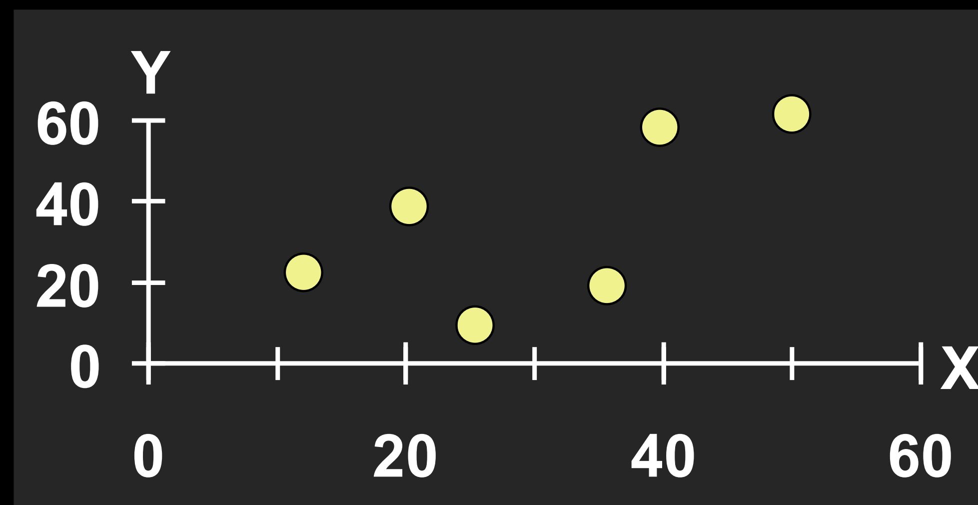
Linear Regression Model



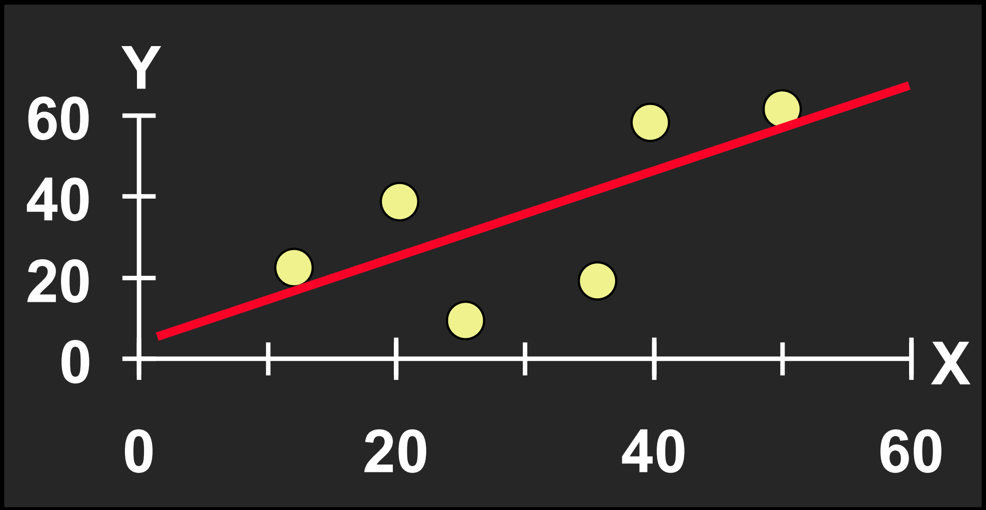
Fitting the Best Line

- Plot all (X_i, Y_i) Pairs



Fitting the Best Line

- Plot all (X_i, Y_i) Pairs
- Draw a line. But how do we know it is best?



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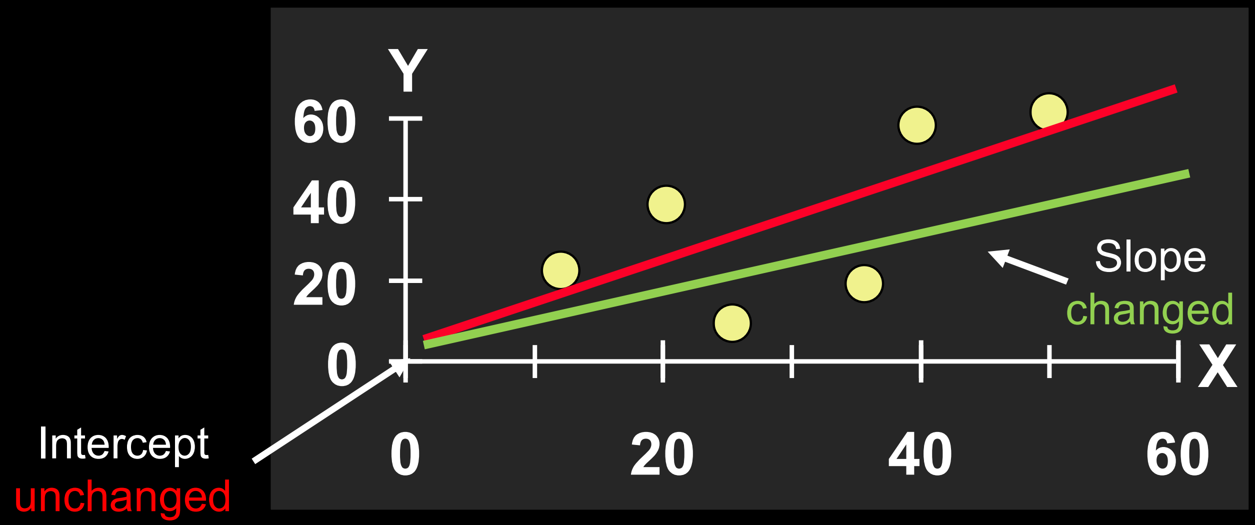

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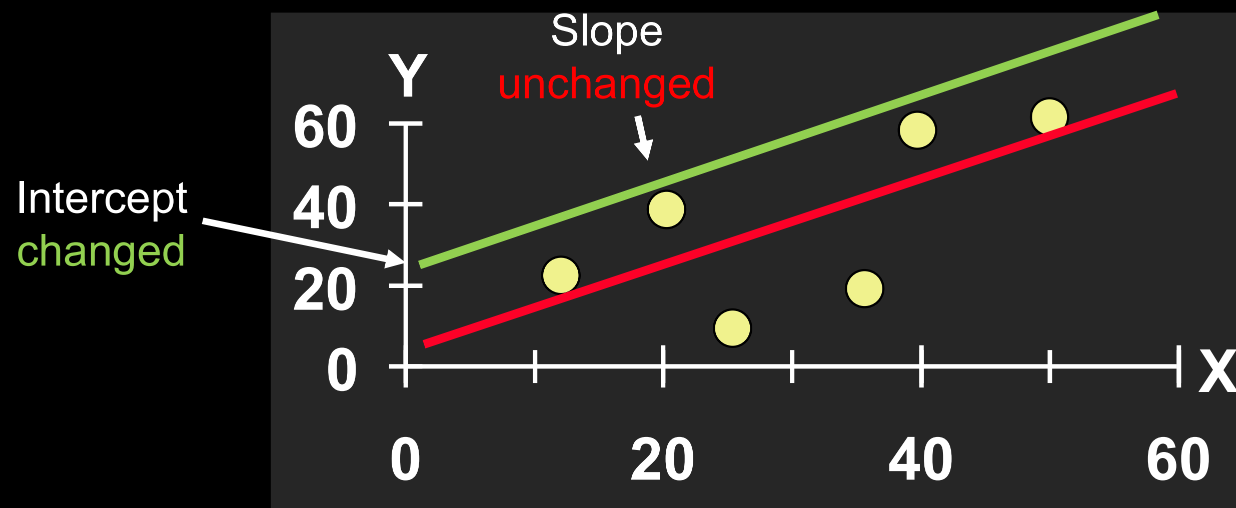

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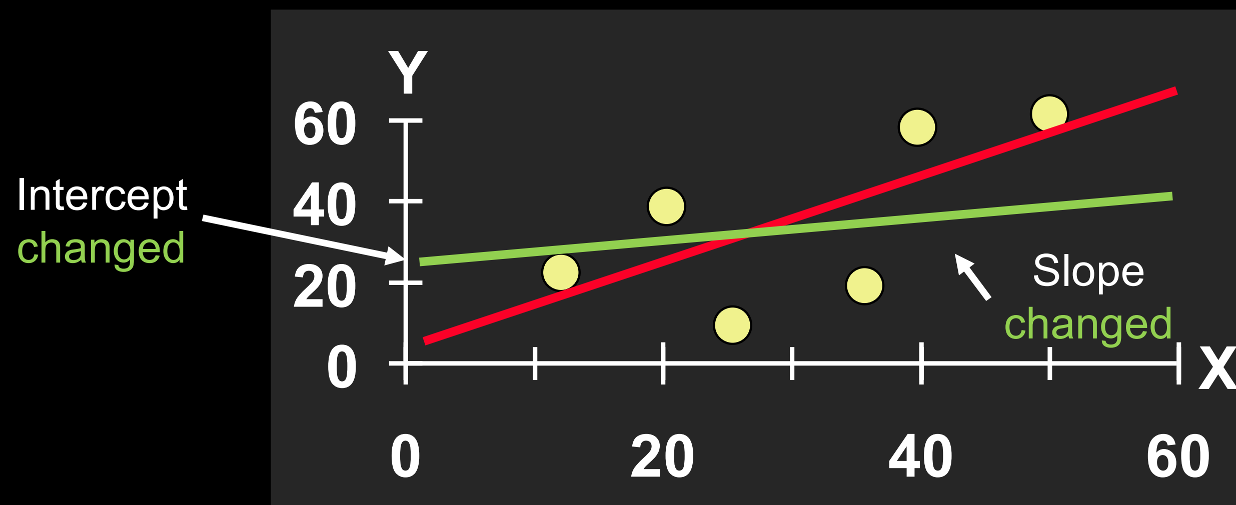


Linear Regression Model

- Relationship between variables is linear function, but with error term

Population Y-Intercept

Population Slope

Random Prediction Error

$$Y_i = b_0 + mX_i + \varepsilon_i$$

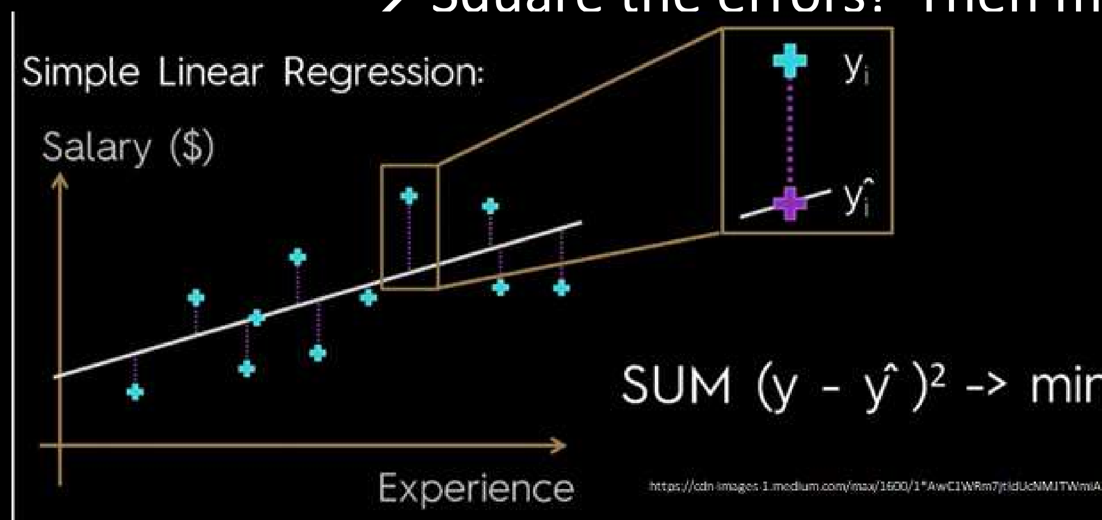
Dependent (response) Variable (e.g., kills)

Independent (explanatory) Variable (e.g., skill level)

Want error as small as possible

Least Squares Line

- Want to minimize difference between actual y and predicted $\hat{y}$
  - Add up ε_i for all observed y's
  - But positive differences offset negative ones
  - (remember when this happened for variance?)
  - → Square the errors! Then minimize (Calculus)

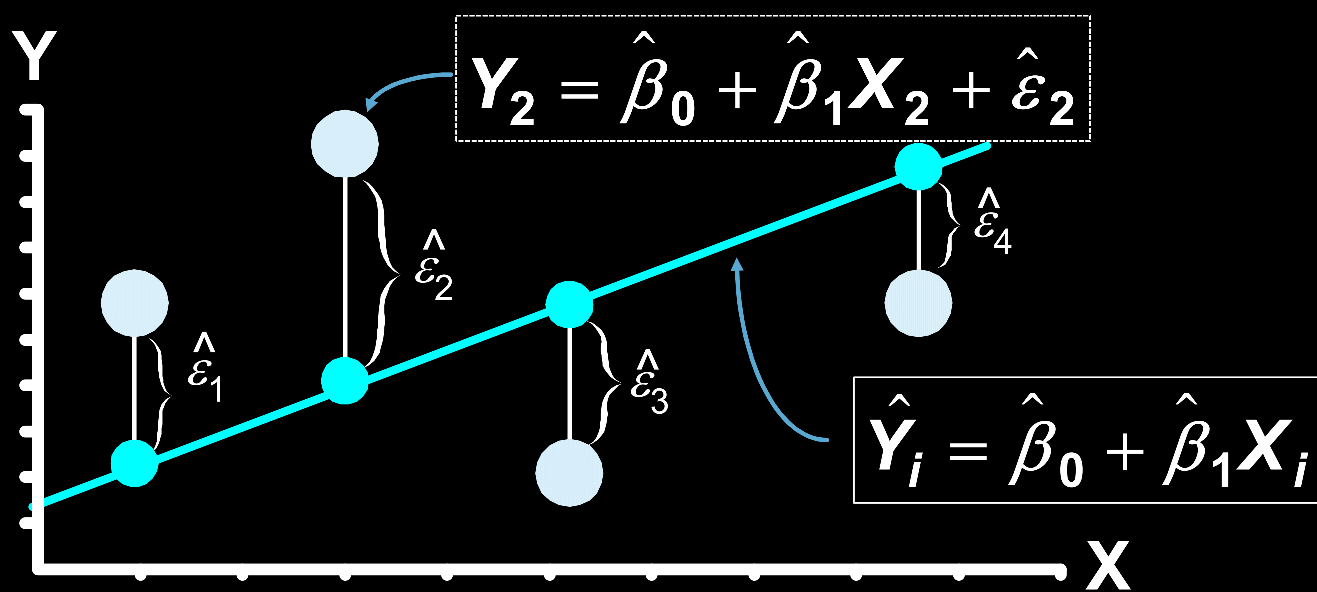


Minimize:
Take derivative
Set to 0 and solve

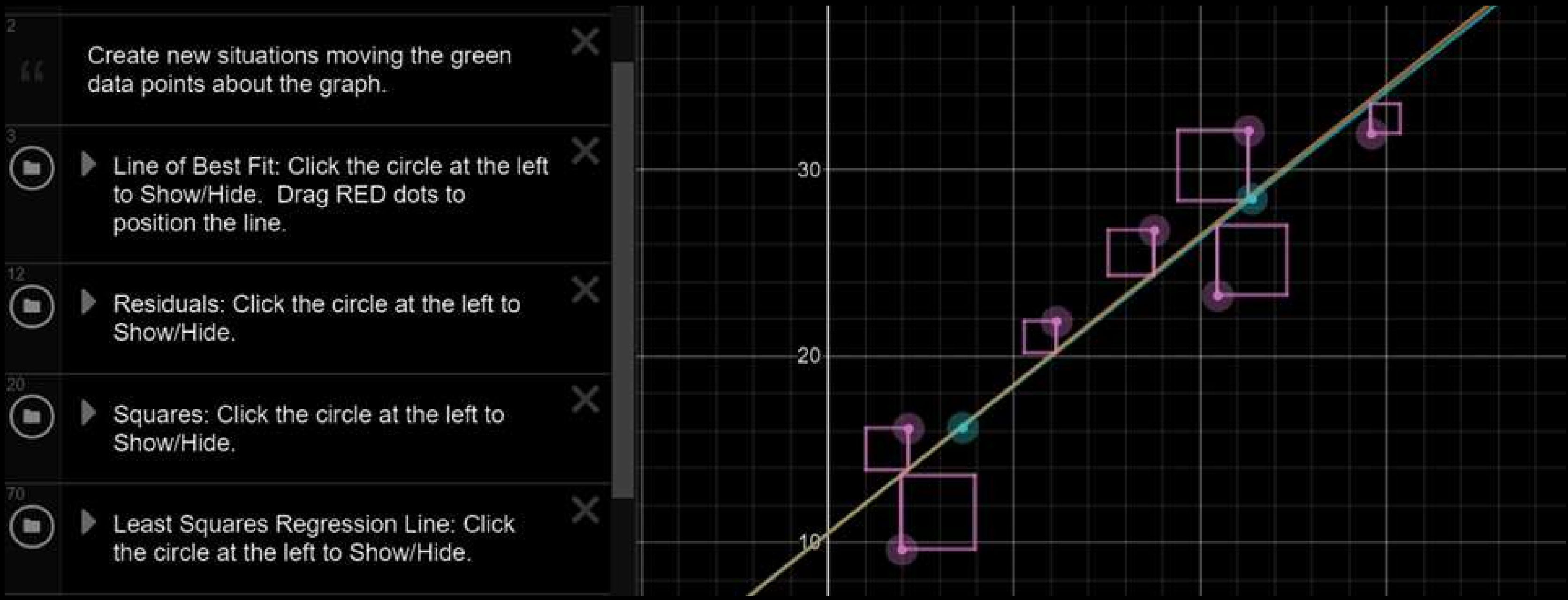
$$\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)^2$$

Least Squares Line Graphically

LS minimizes $\sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \hat{\varepsilon}_1^2 + \hat{\varepsilon}_2^2 + \hat{\varepsilon}_3^2 + \hat{\varepsilon}_4^2$



Least Squares Line Graphically

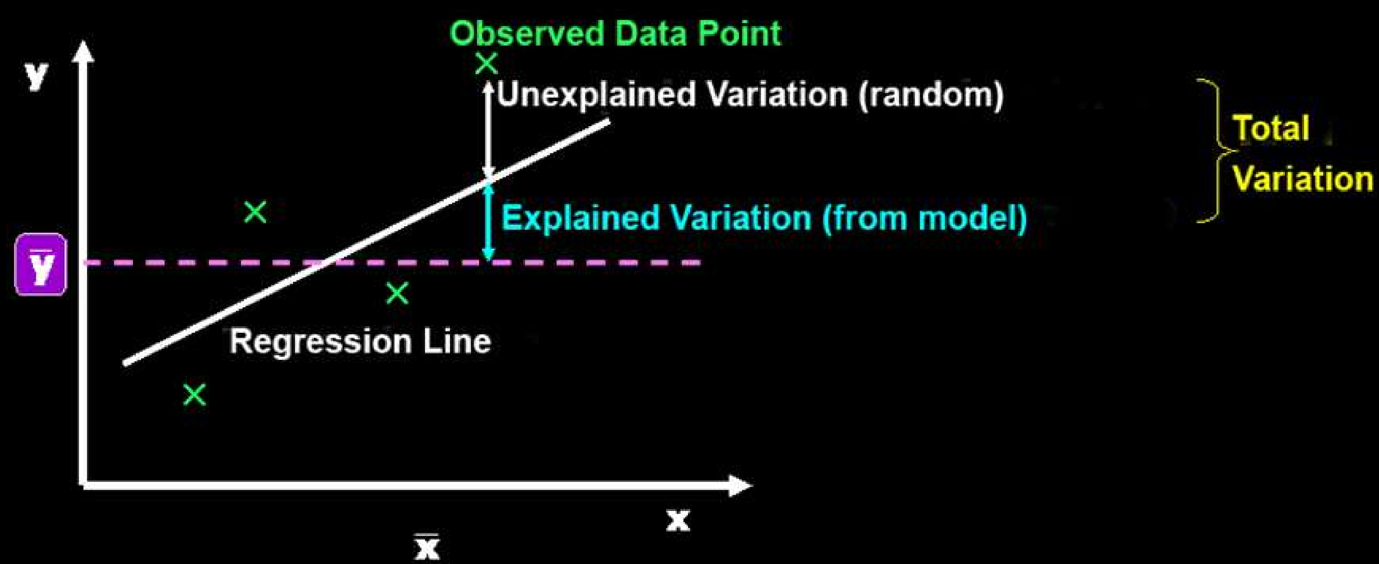


https://www.desmos.com/calculator/zvrc4lg3cr

Outline

- Introduction (done)
- Simple Linear Regression (done)
- Measures of Variation (next)
  - Coefficient of Determination
  - Correlation
- Misc

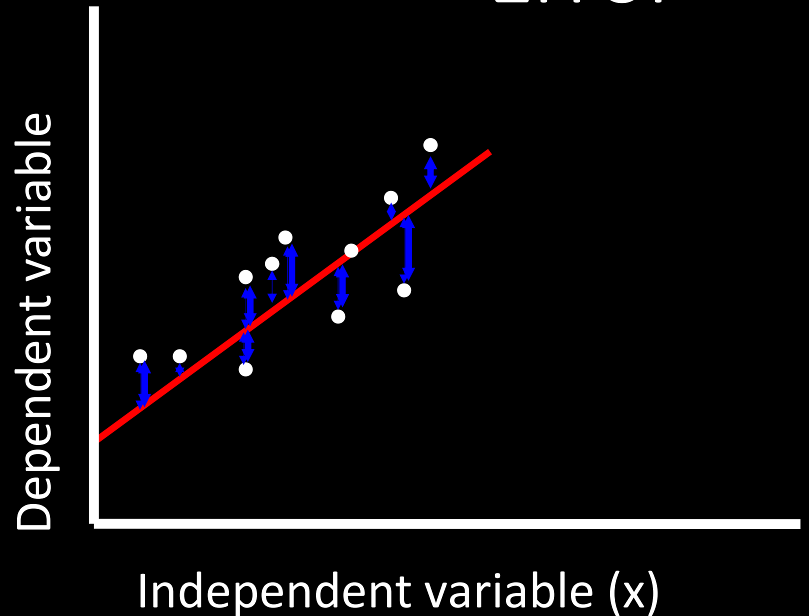
Measures of Variation



- Several sources of variation in y
  - Error in prediction (unexplained)
  - Variation from model (explained)

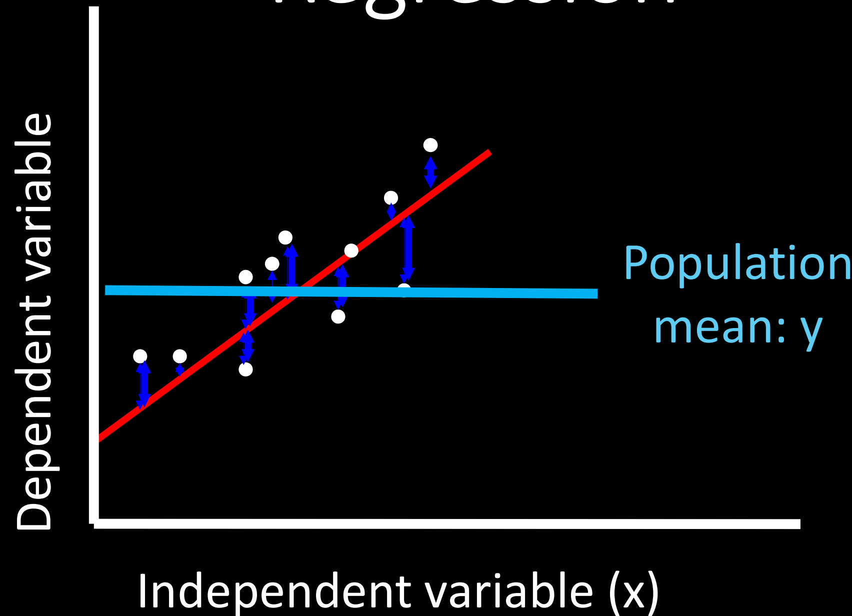
Break this down (next)

Sum of Squares of Error



- Least squares regression line with lowest total sum of squared prediction errors
- Sum of Squares of Error, or SSE
- Measure of unexplained variation

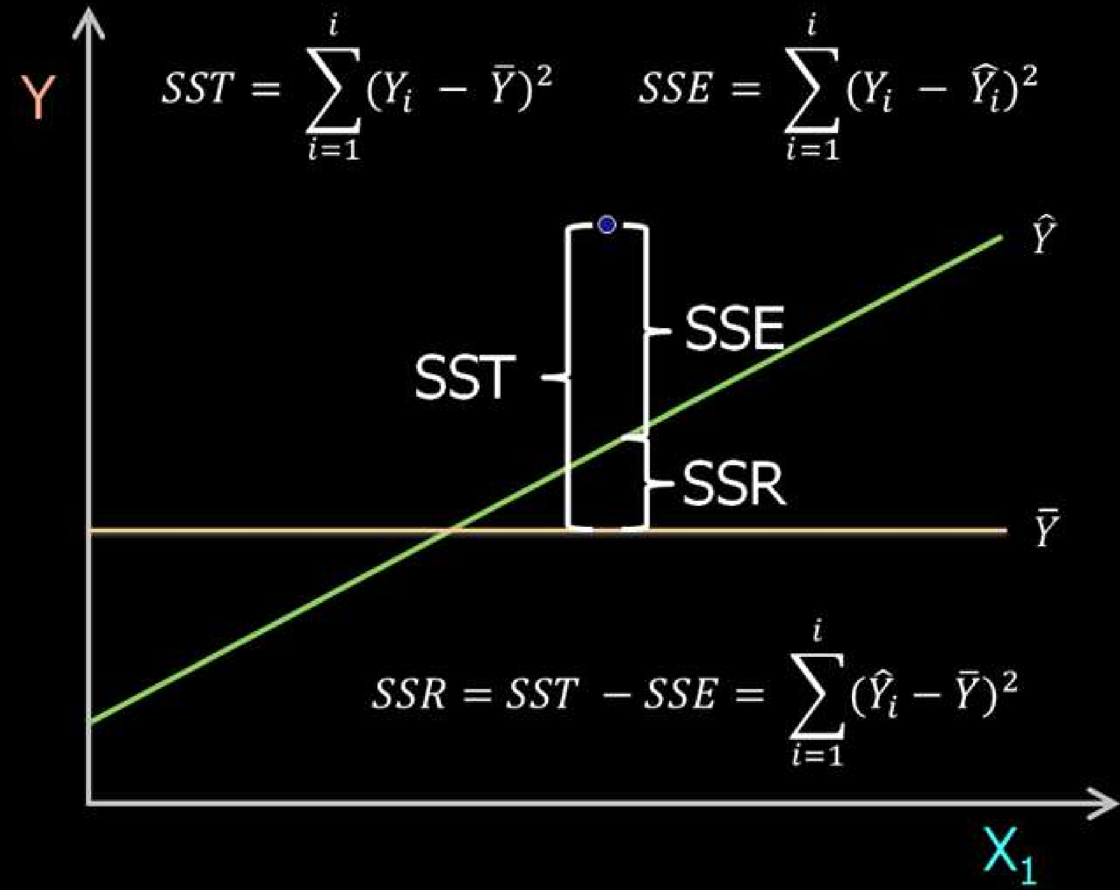
Sum of Squares Regression



- Differences between prediction and population mean
  - Gets at variation due to X & Y
- Sum of Squares Regression, or SSR
- Measure of explained variation

Sum of Squares
Total

- Total Sum of Squares, or SST = SSR + SSE



Coefficient of Determination

- Proportion of total variation (SST) explained by the regression (SSR) is known as the Coefficient of Determination (R^2)

$$R^2 = \frac{SSR}{SST} = 1 - \frac{SSE}{SST}$$

- Ranges from 0 to 1 (often said as a percent)

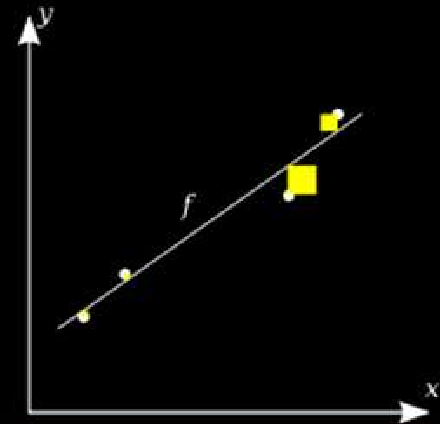
  1 – regression explains all of variation

  0 – regression explains none of variation

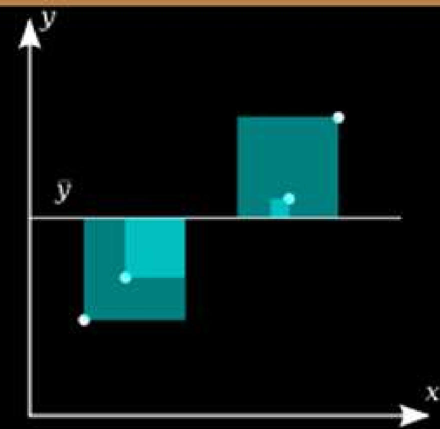
Coefficient of Determination – Visual Representation

GA

$R^2 = 1 -$

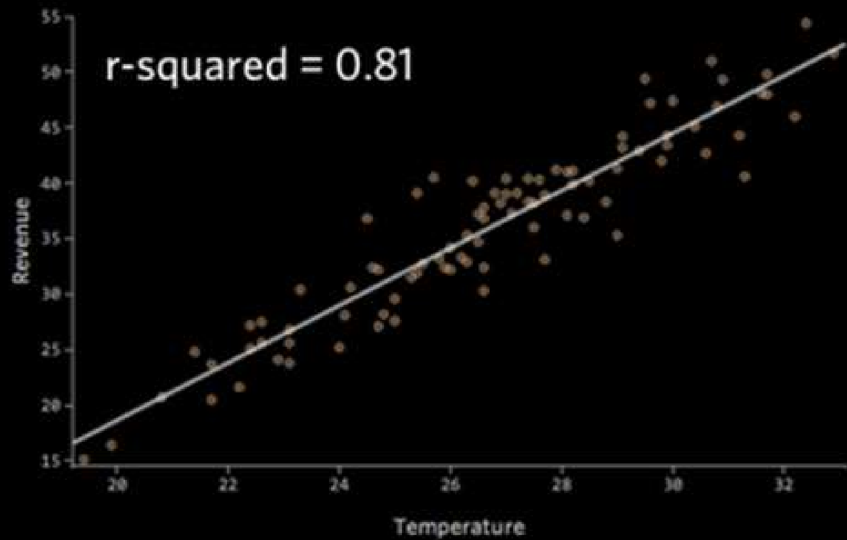


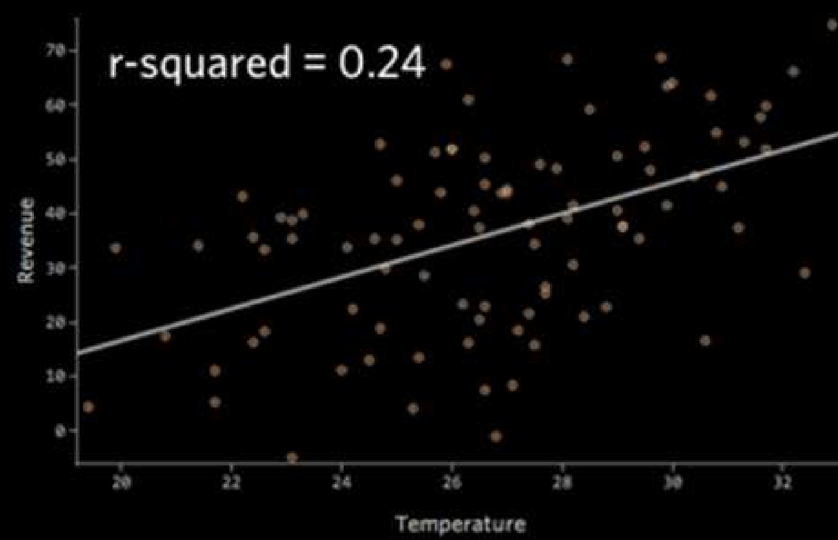
Variation in observed data model cannot explain (error)



Total variation in observed data

Coefficient of Determination Example

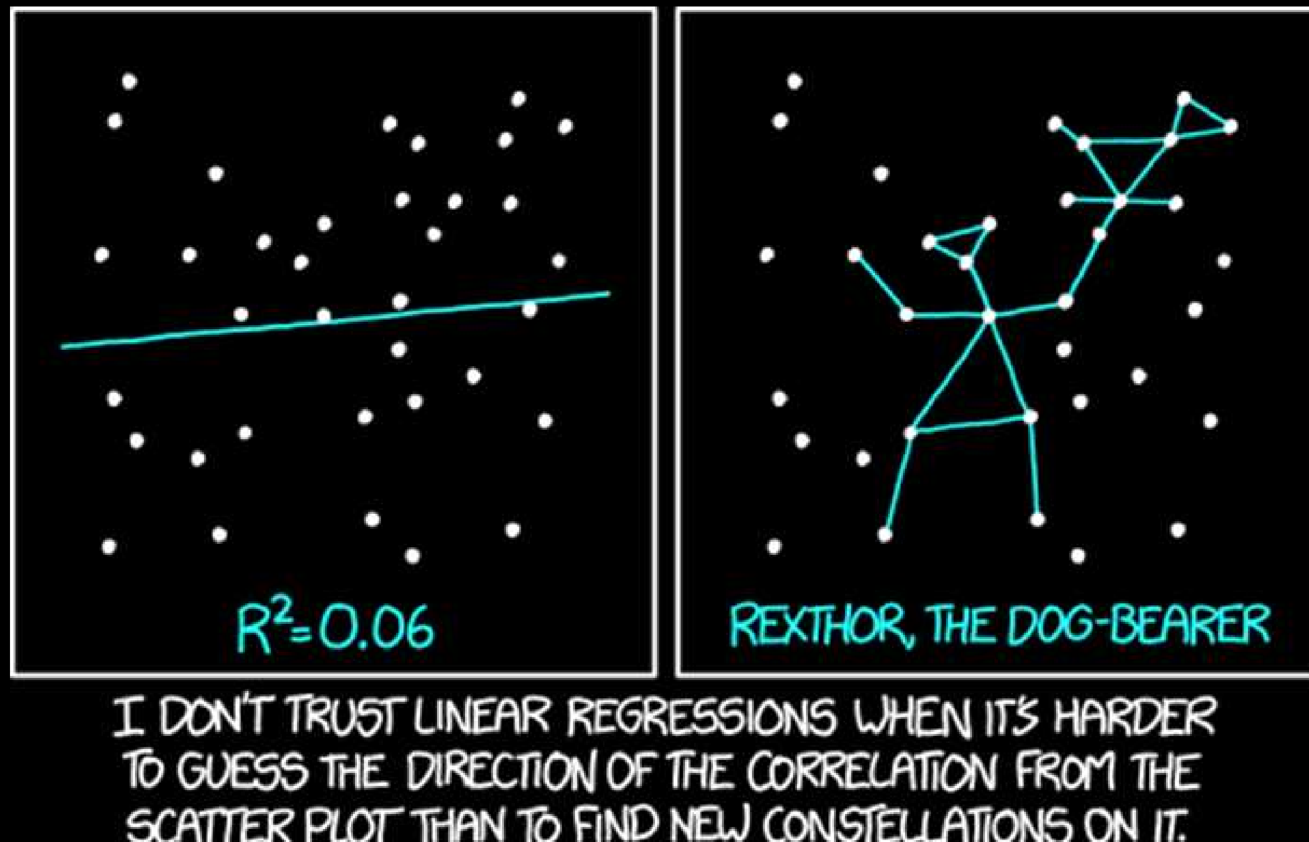




- How "good" is regression model? Roughly:

$0.8 \le R^2 \le 1$	strong
$0.5 \le R^2 < 0.8$	medium
$0 \le R^2 < 0.5$	weak

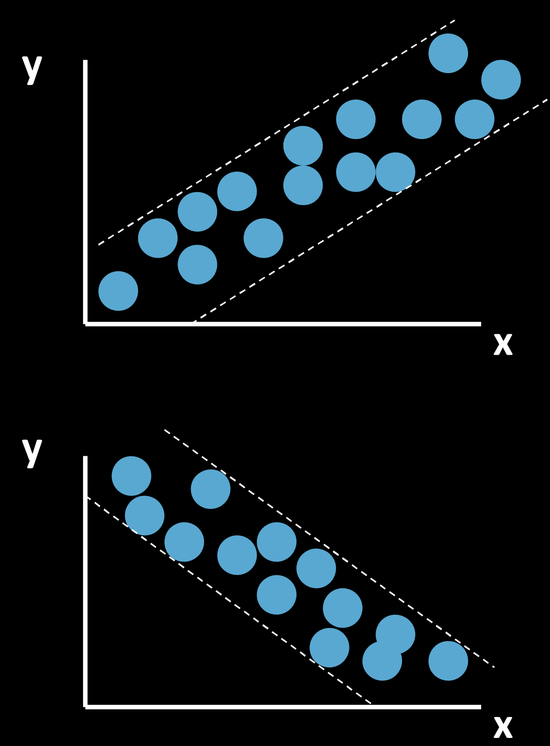
How “good” is the Regression Model?



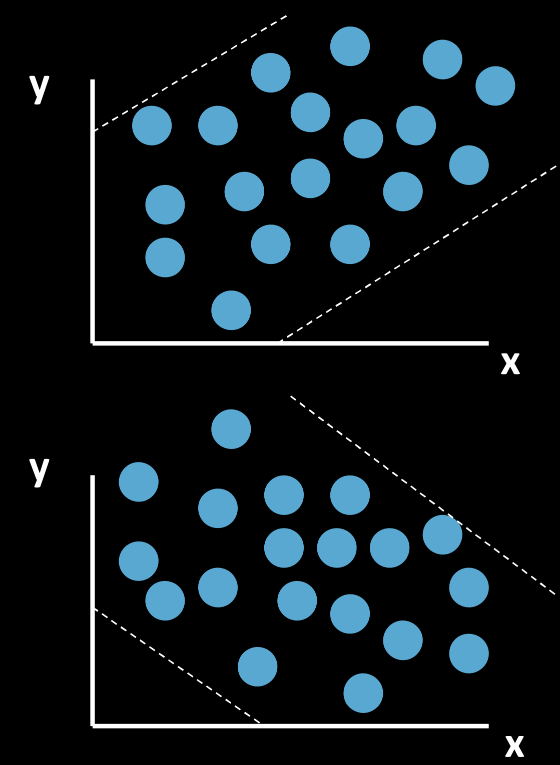
https://xkcd.com/1725/

Relationships Between X & Y

Strong relationships



Weak relationships



Relationship Strength and Direction – Correlation

- Correlation measures strength and direction of linear relationship
  - -1 perfect neg. to +1 perfect pos.
  - Sign is same as regression slope
  - Denoted R. Why? R = $\sqrt{R^2}$

Pearson's Correlation Coefficient

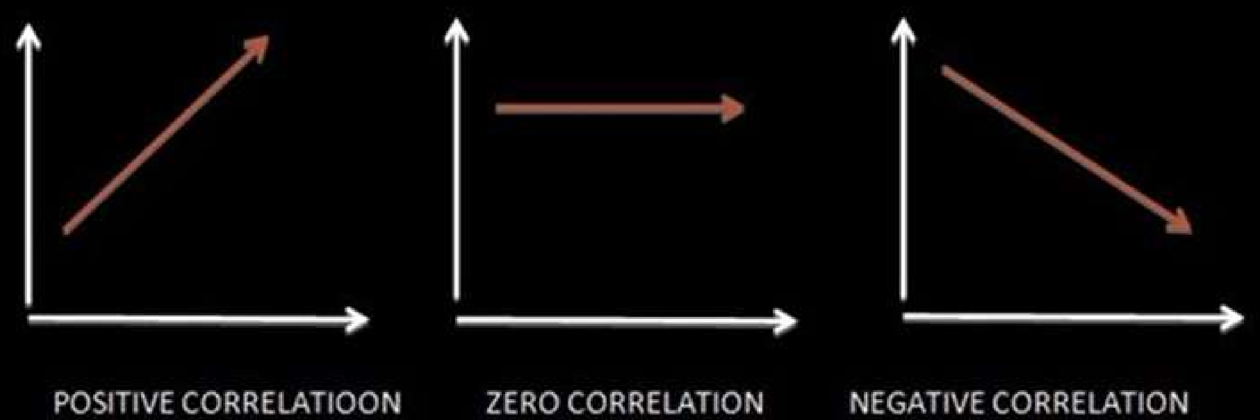
$$r = \frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sqrt{\sum(X-\overline{X})^2}\sqrt{(Y-\overline{Y})^2}}$$

Vary together

Vary Separately

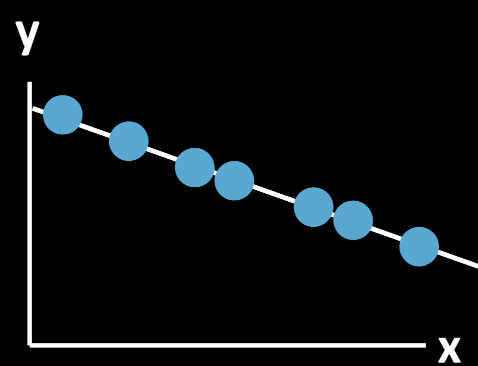
Where, $\overline{X}$ = mean of X variable

$\overline{Y}$ = mean of Y variable

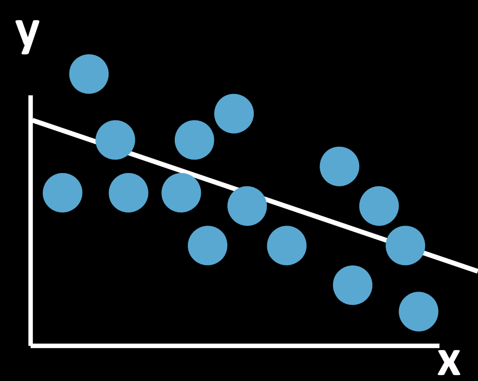


https://www.mbaskool.com/2013_images/stories/dec_images/pearson coeff bcon.jpg

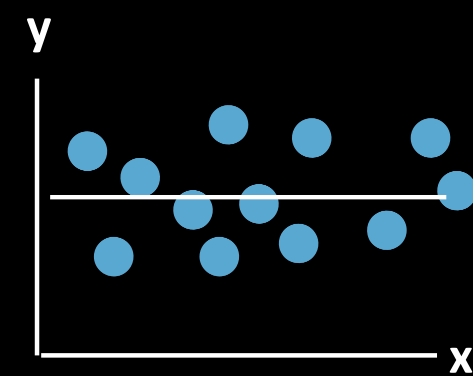
Correlation Examples



r = -1

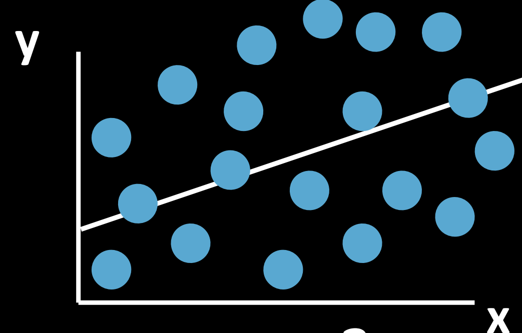


r = -.6

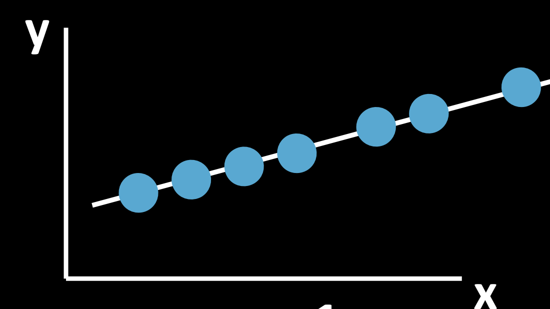


r = 0

Also, a measure of "Effect Size"



r = +.3



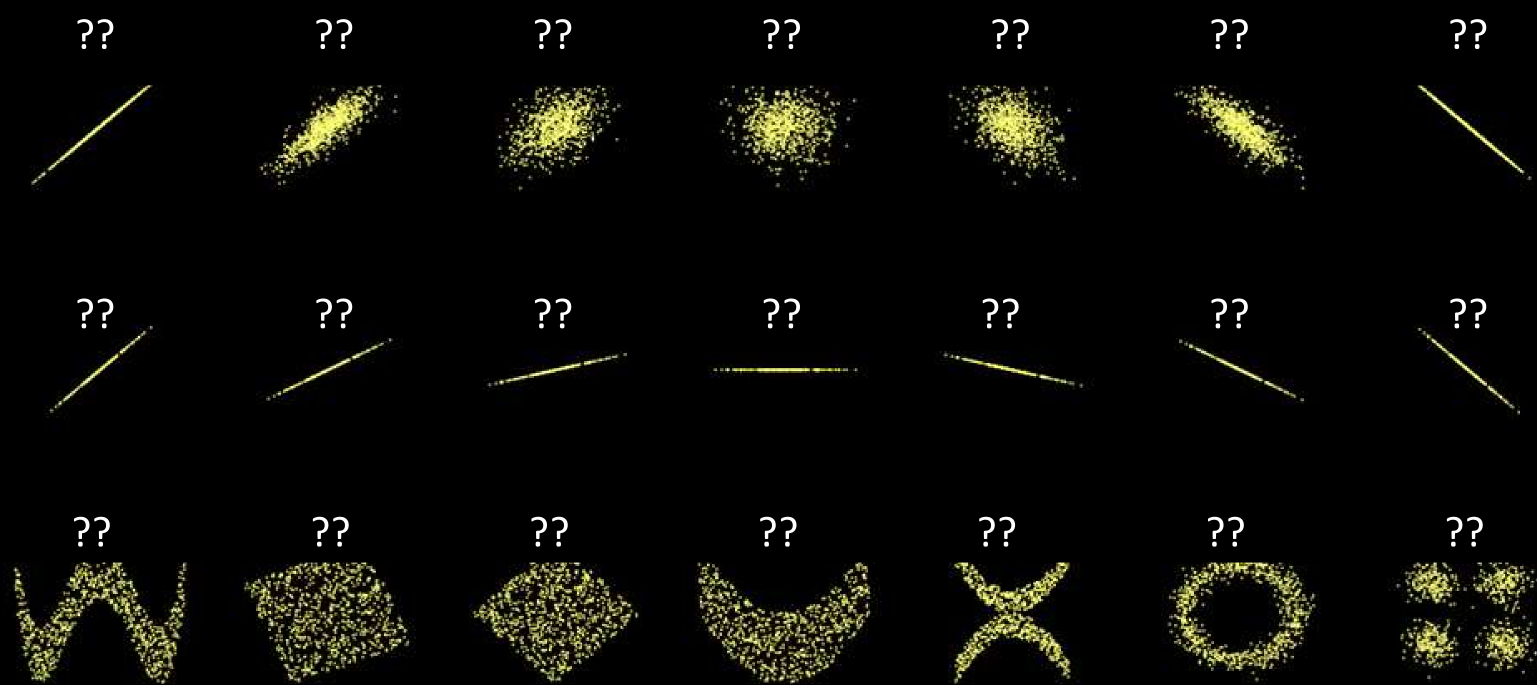
r = +1

Breakout 7

- Introduction
  - If needed ... Introduce yourselves!
  - **Icebreaker**: What game are you looking forward to playing this summer?
- Groupwork
  - Think, discuss, write down – email answers
- Correlation
  - Consider scatterplots
  - Estimate correlation

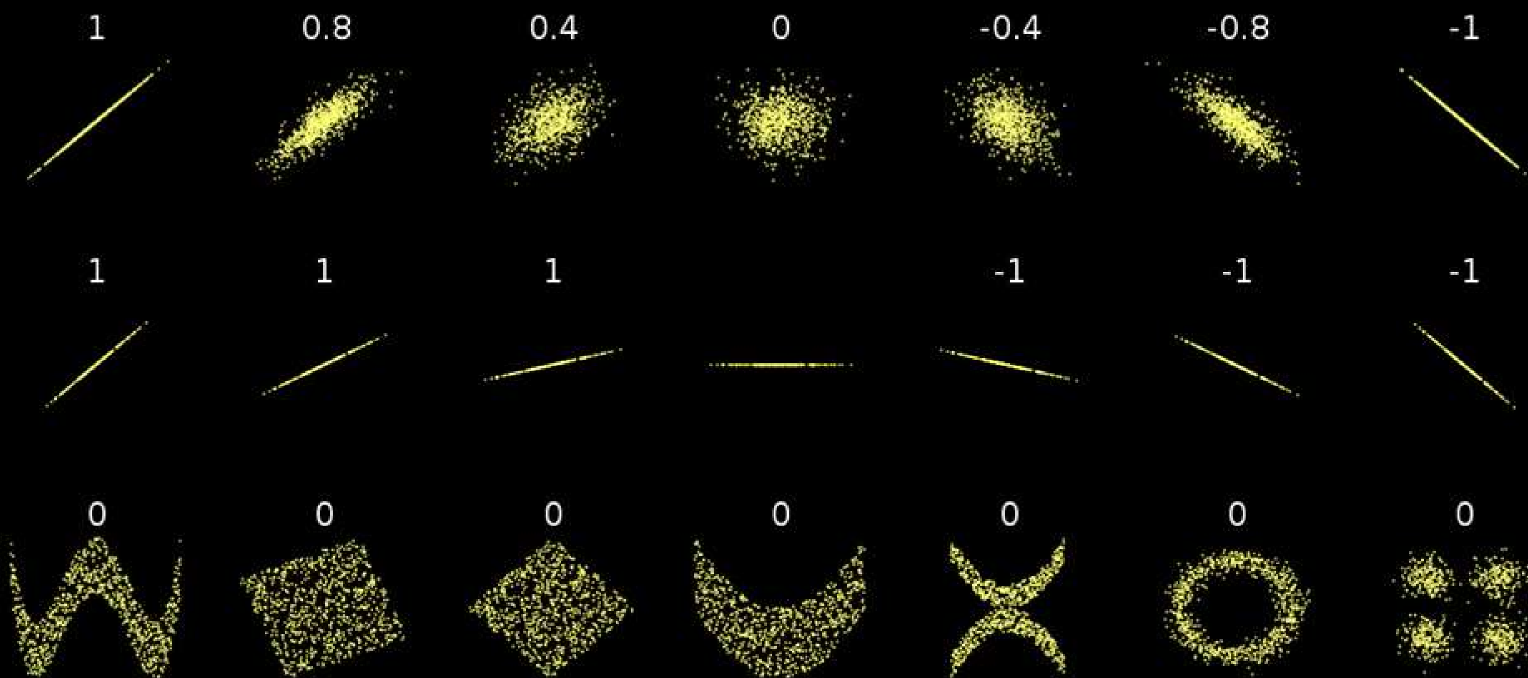
https://web.cs.wpi.edu/~imgd2905/d21/breakout/breakout-7.html

Correlation Examples



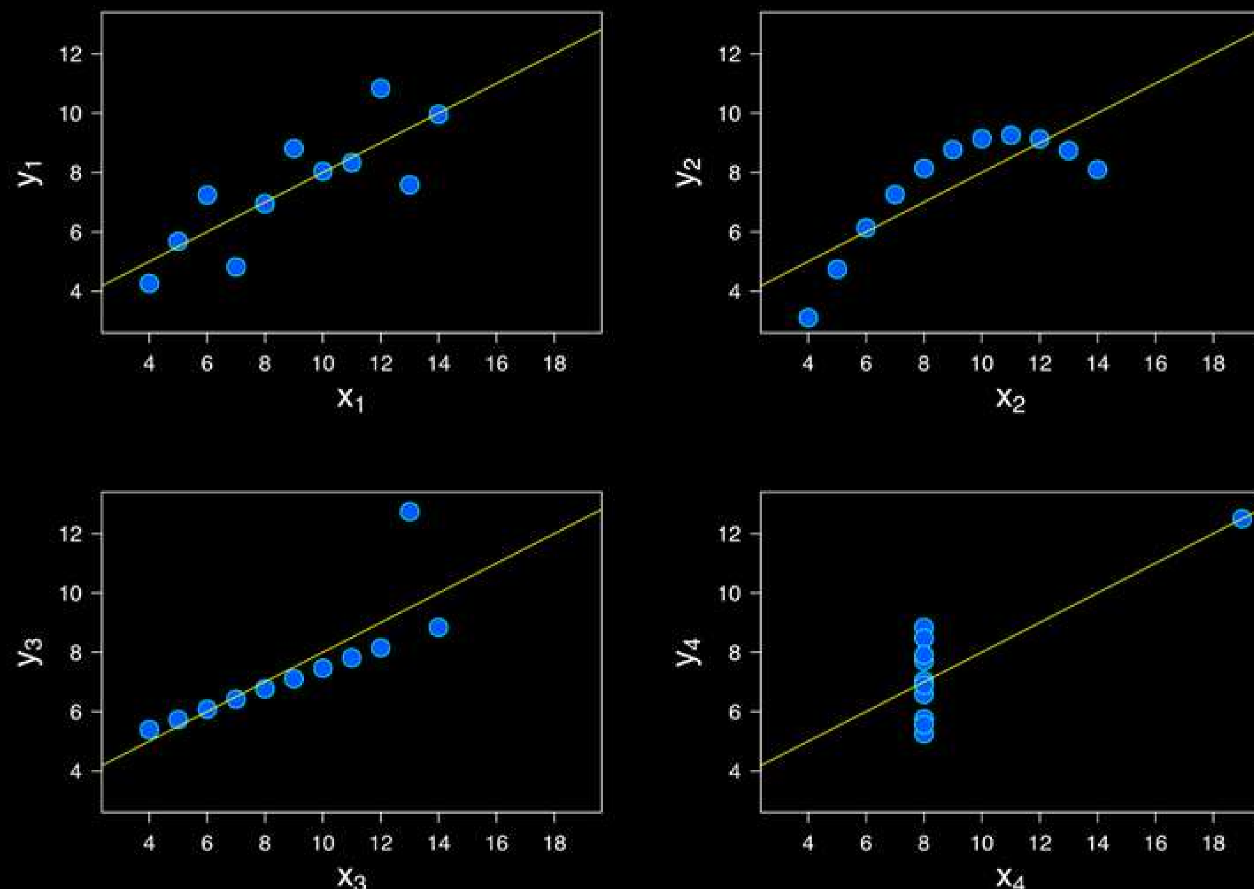
https://upload.wikimedia.org/wikipedia/commons/thumb/d/d4/Correlation_examples2.svg/1200px-Correlation_examples2.svg.png

Correlation Examples

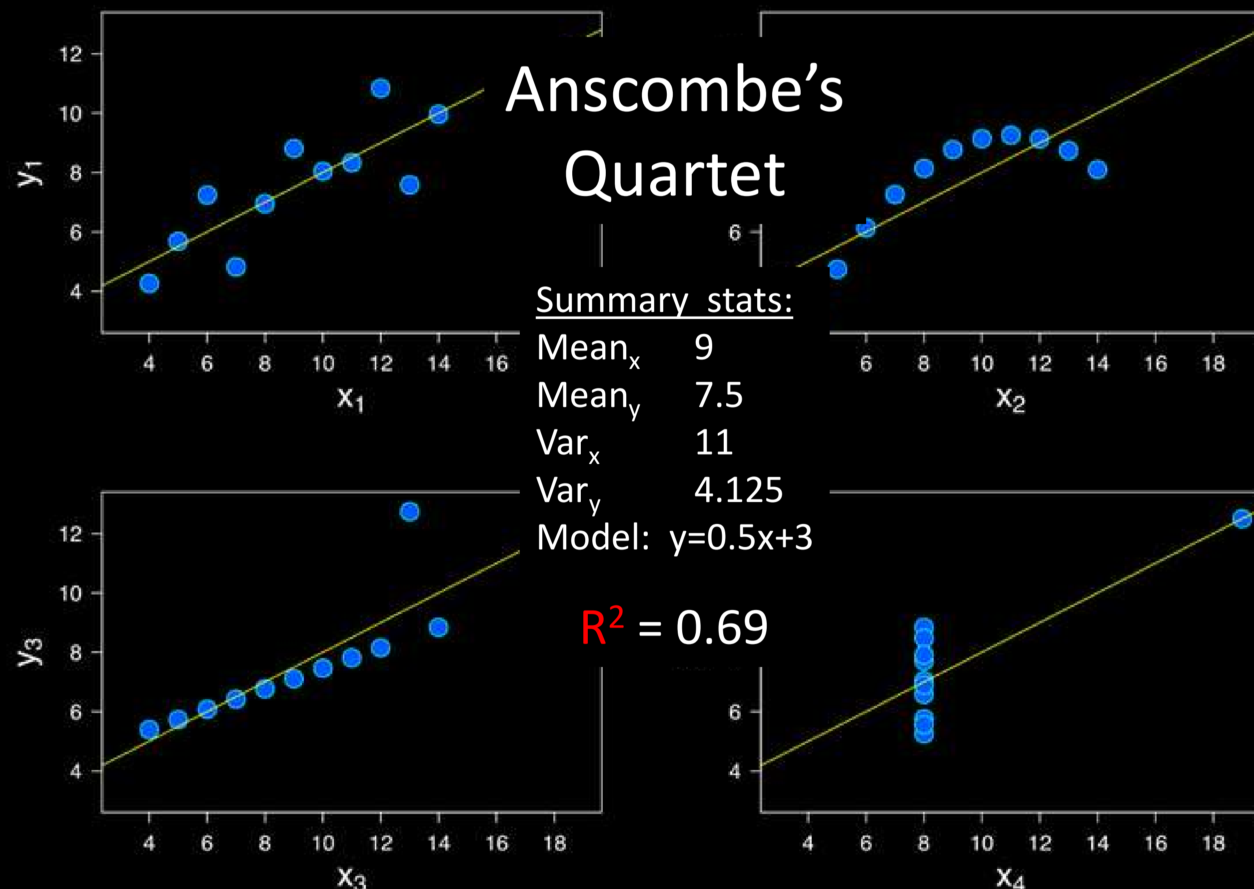


https://upload.wikimedia.org/wikipedia/commons/thumb/d/d4/Correlation_examples2.svg/1200px-Correlation_examples2.svg.png

Correlation Examples



Correlation Examples



Anscombe's Quartet

Summary stats:

$Mean_x$ 9

$Mean_y$ 7.5

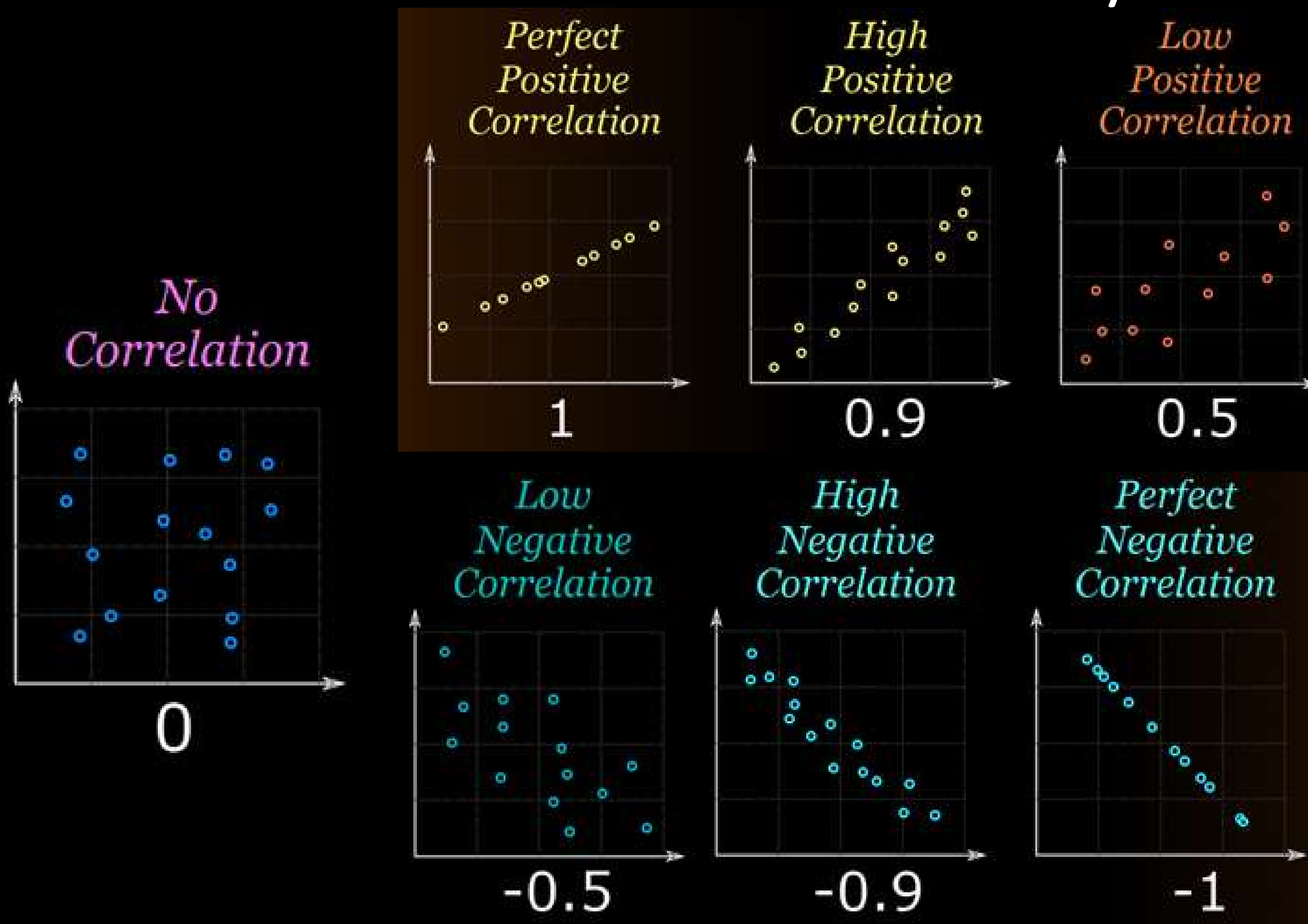
Var_x 11

Var_y 4.125

Model: y=0.5x+3

$R^2 = 0.69$

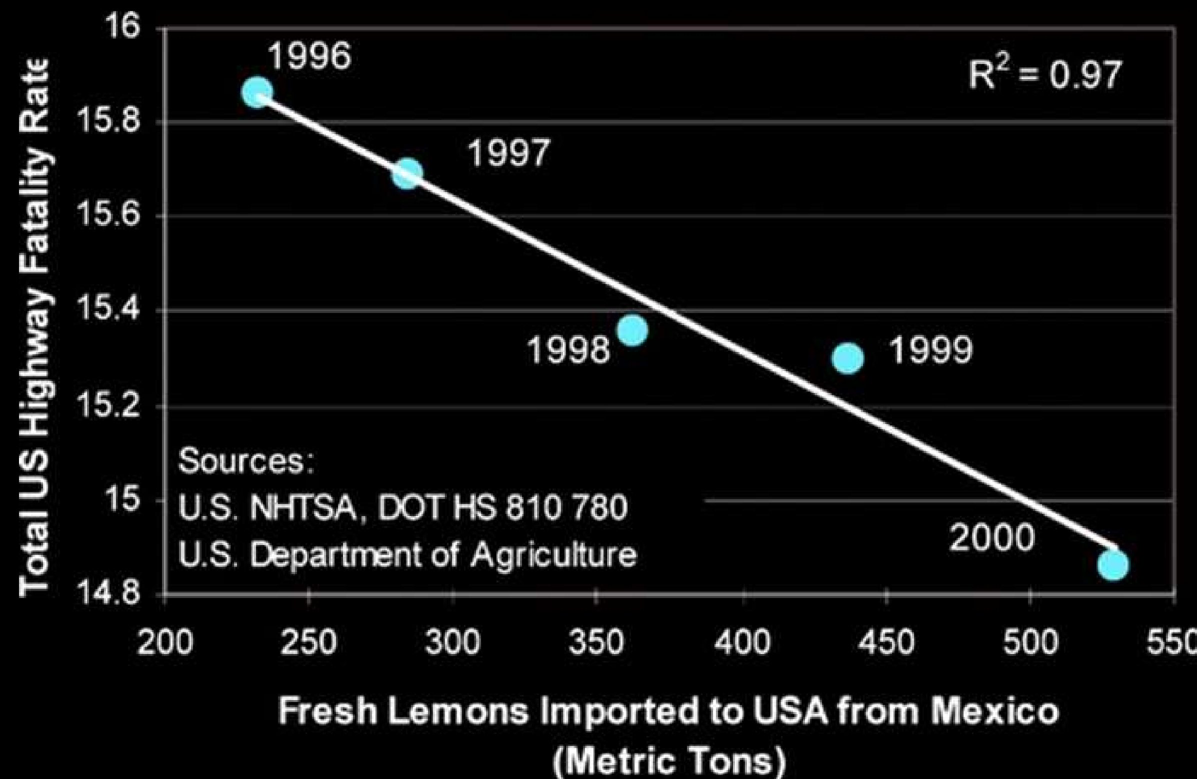
Correlation Summary



Correlation is not Causation

Buying sunglasses *causes* people to buy ice cream?

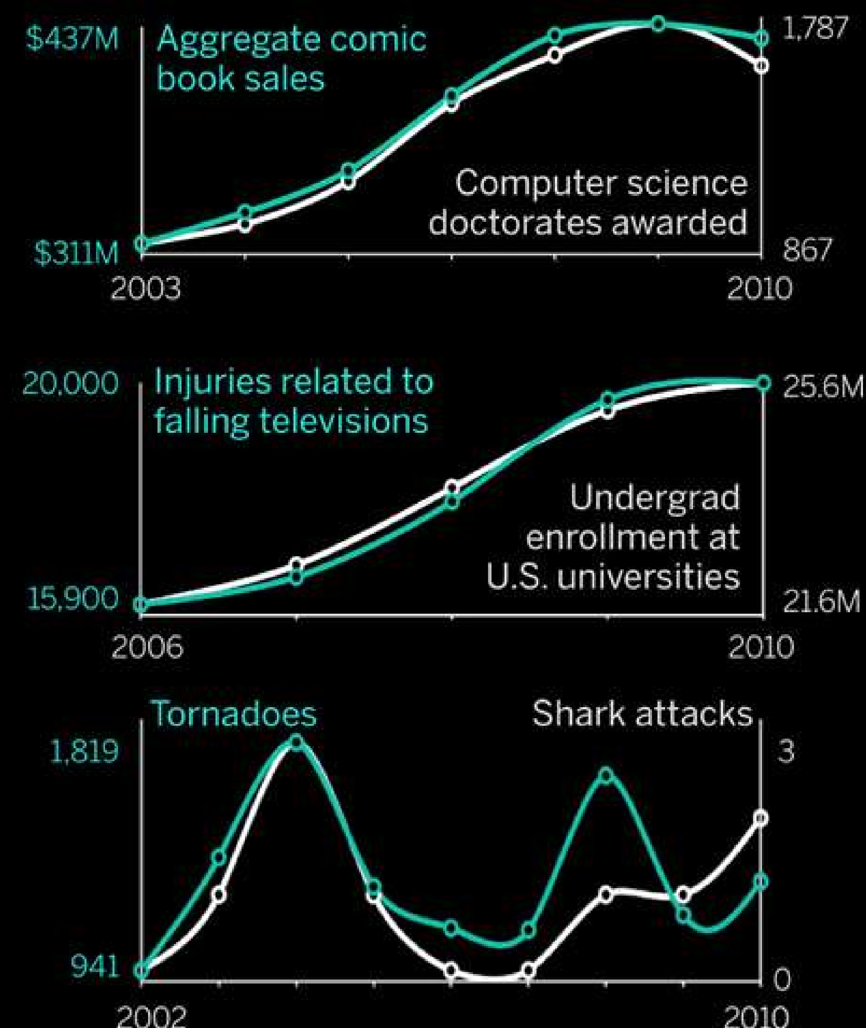
Correlation is not Causation



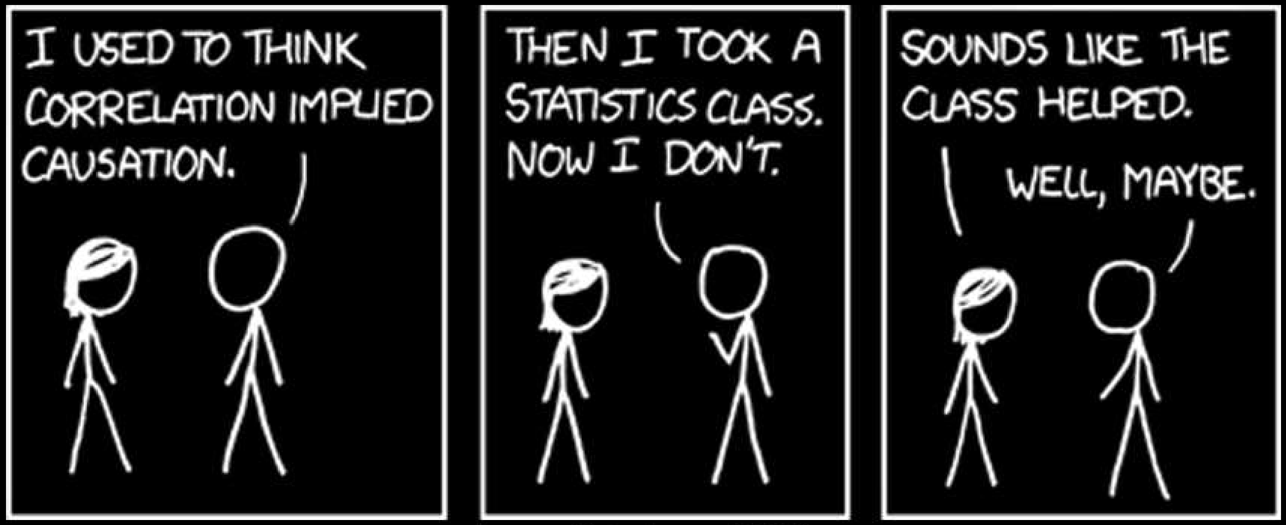
Importing lemons causes fewer highway fatalities?

Correlation is not Causation

https://science.sciencemag.org/content/sci/348/6238/980.2/F1.large.jpg?width=800&height=600&carousel=1



Correlation is not Causation



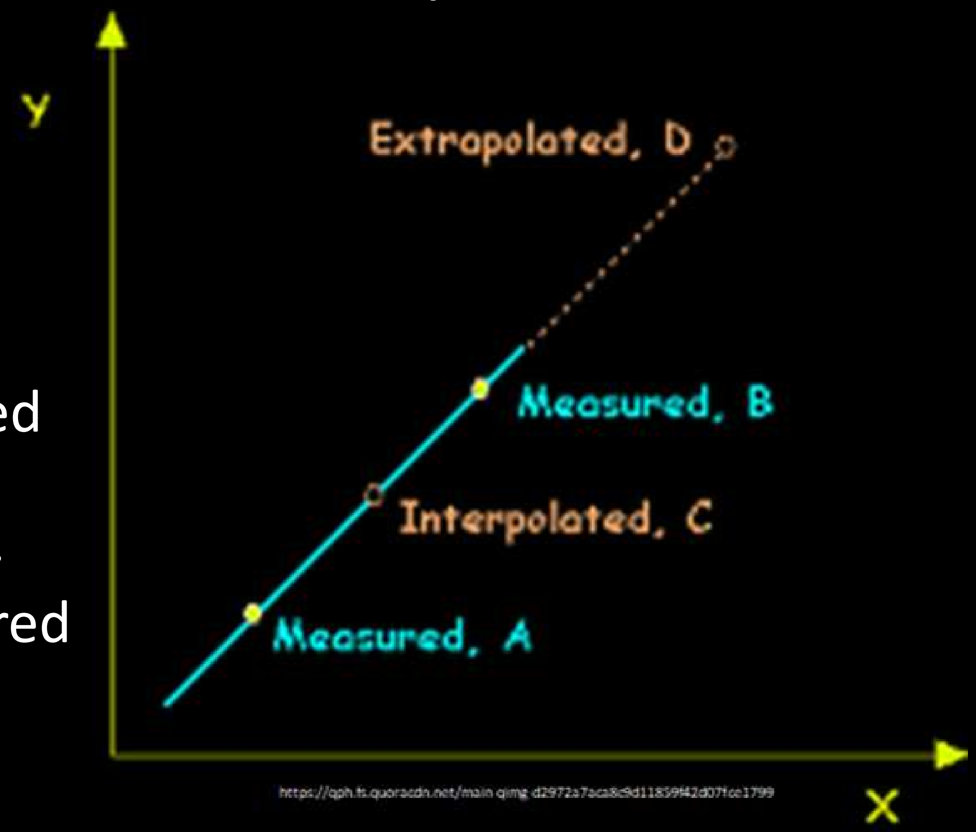
https://xkcd.com/552/

Outline

- Introduction (done)
- Simple Linear Regression (done)
- Measures of Variation (done)
- Misc (next)
  - Extrapolation and Interpolation
  - Confidence Intervals
  - Model fitting

Extrapolation versus Interpolation

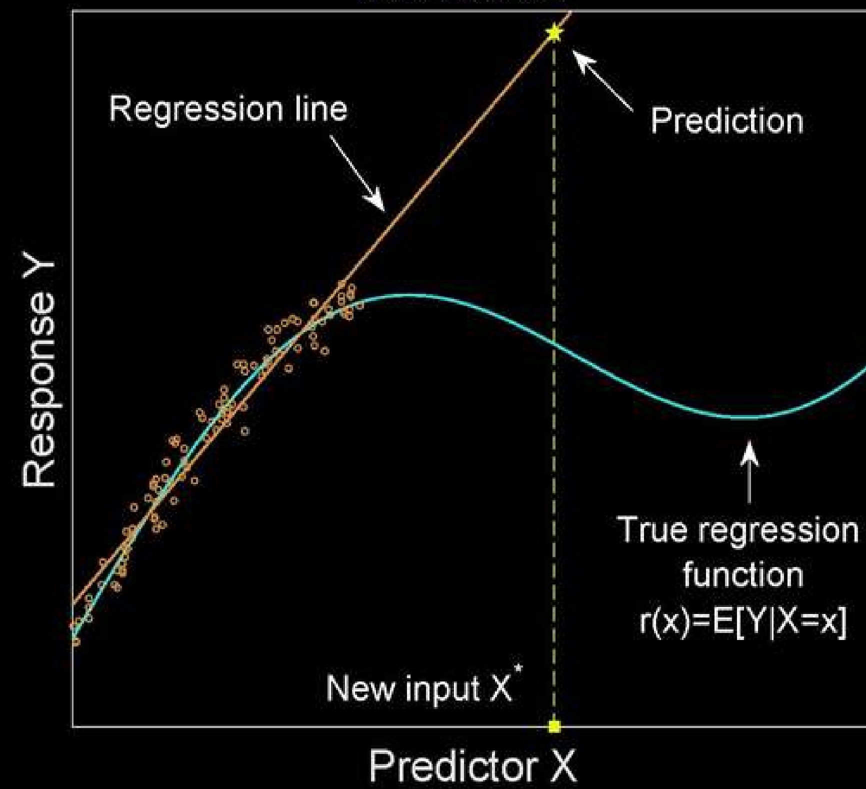
- Prediction
  - Interpolation – within measured X-range
  - Extrapolation – outside measured X-range



https://qph.fs.quoracdn.net/main-qimg-d2972a7aca8c9d11859f42d07fce1799

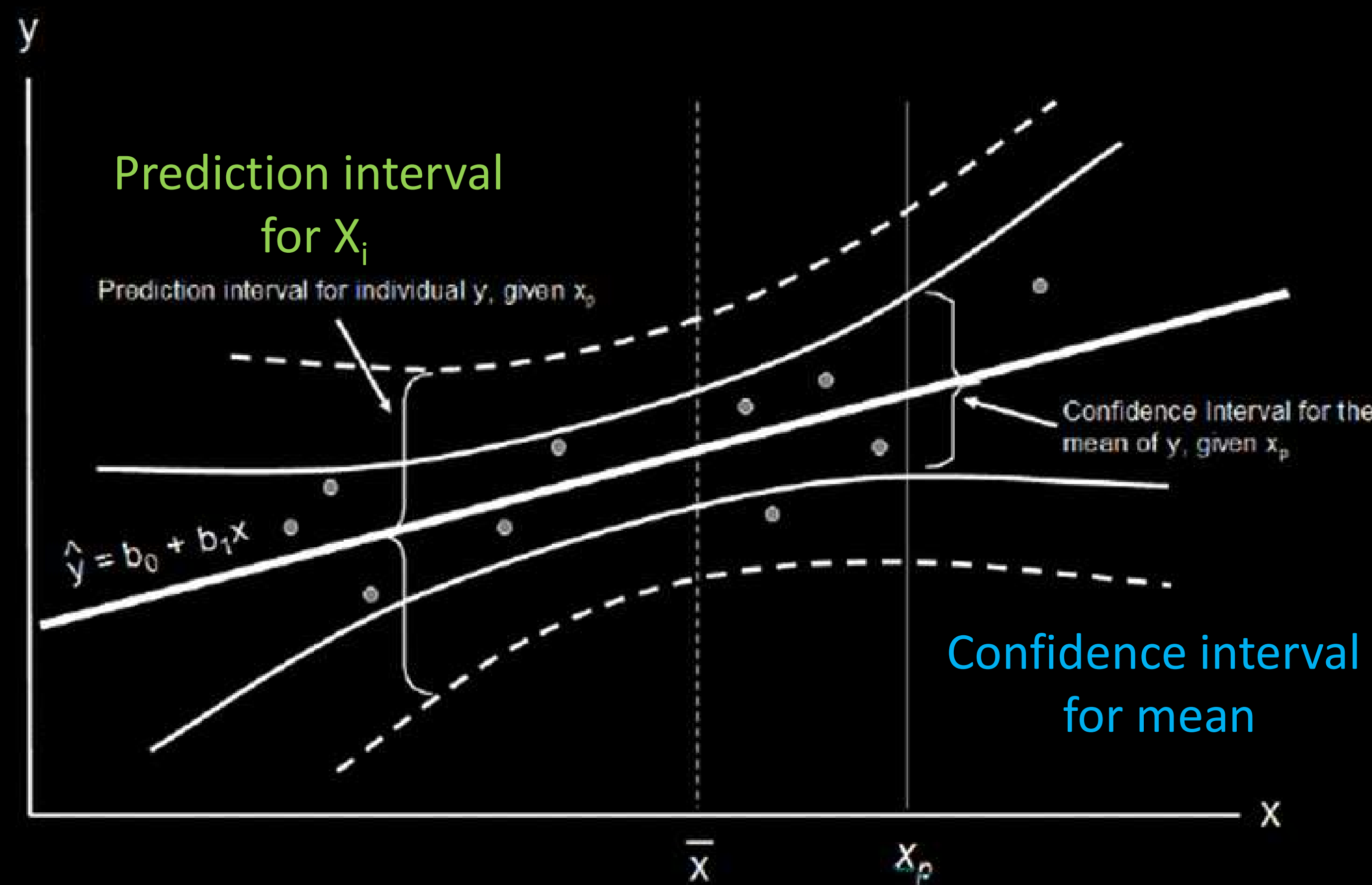
Be Careful When Extrapolating

https://i.stack.imgur.com/3Ab7e.jpg

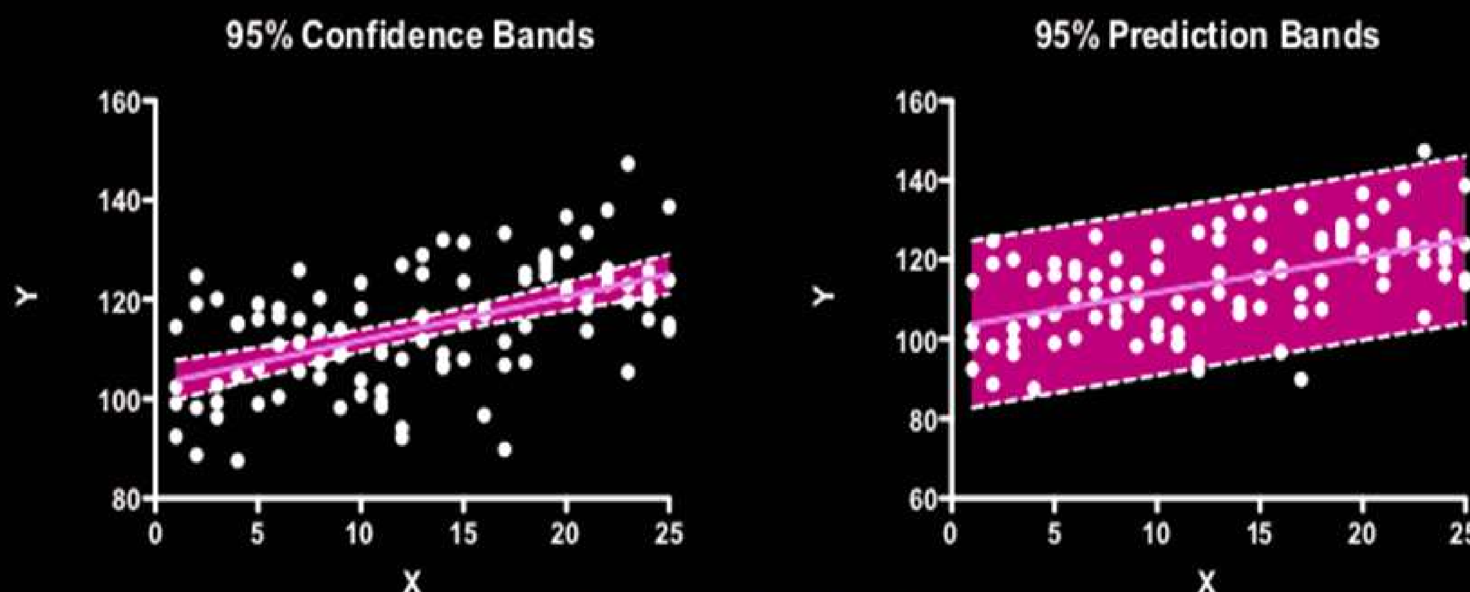


If extrapolate, make sure have reason to assume model ***continues***

Prediction and Confidence Intervals (1 of 2)



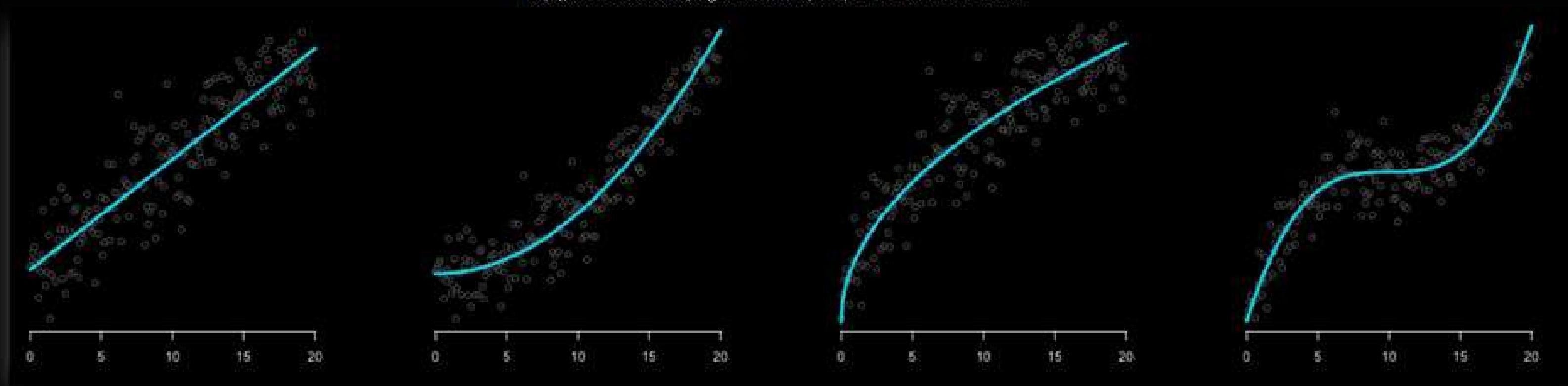
Prediction and Confidence Intervals (2 of 2)



https://www.graphpad.com/guides/prism/7/curve-fitting/reg_mostpointsareoutsideconfidencebands.png

Beyond Simple Linear Regression

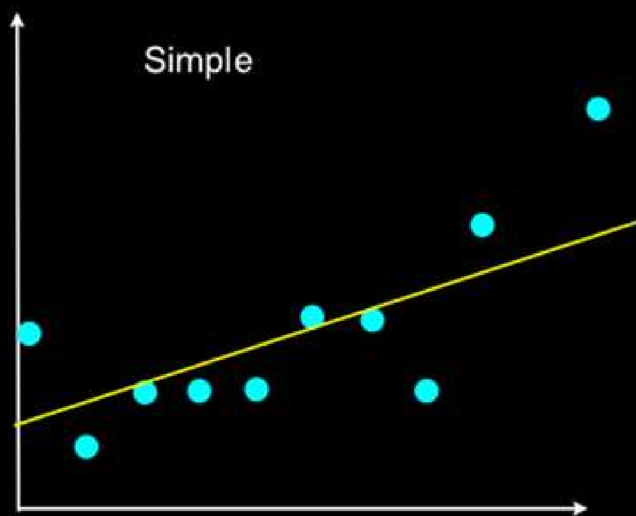
https://medium.freecodecamp.org/learn-how-to-improve-your-linear-models-8294bfa8a731



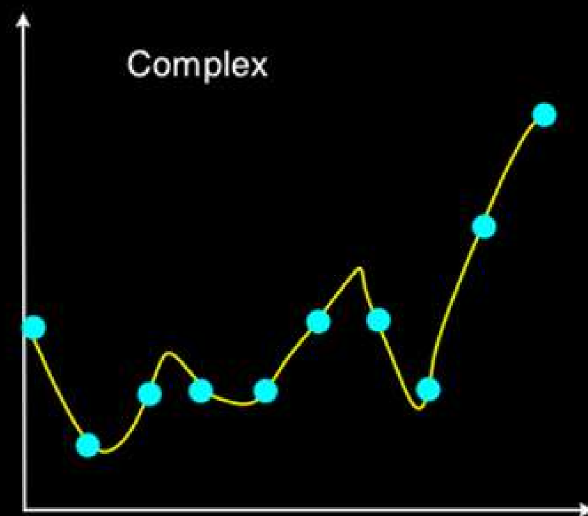
- Multiple regression – more parameters beyond just X
  - Book Chapter 11
- More complex models – beyond just

$Y = mX + b$

More Complex Models



$y = 12x + 9$



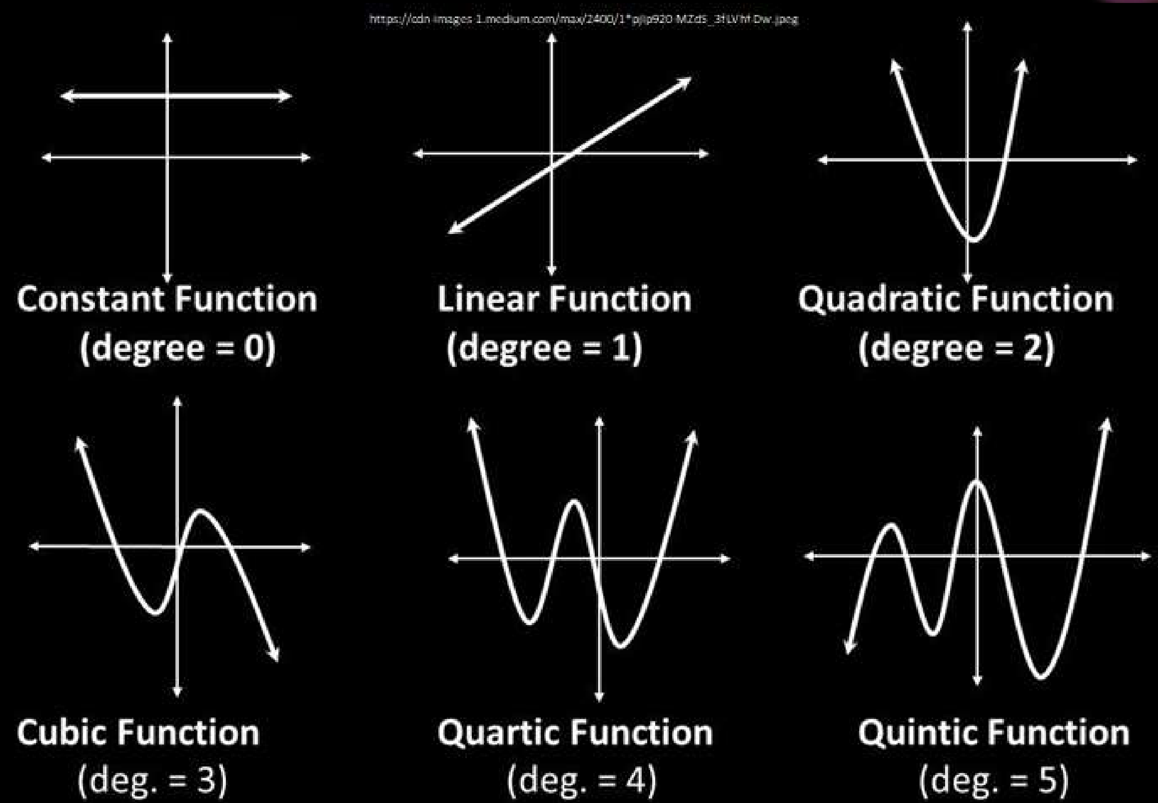
$y = 18x^4 + 13x^3 - 9x^2 + 3x + 20$

- Higher order polynomial model has less error

→ A “perfect” fit (no error)

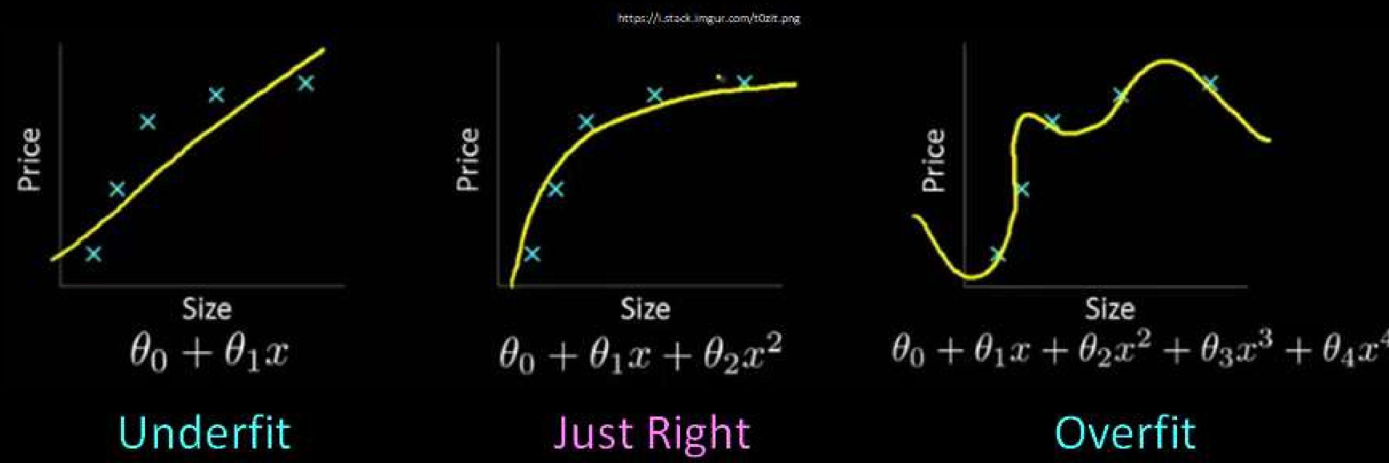
- How does a polynomial do this?

Graphs of Polynomial Functions



Higher degree, more potential "wiggles"
But should you use?

Underfit and Overfit



- Overfit analysis matches data too closely, more parameters than justified
- Underfit analysis does not adequately match since parameters are missing

→ Both models fit well, but don't *predict* well (i.e., non-observed values)

- Just right – fit data well "enough" with as few parameters as possible (*parsimonious* - desired level of prediction with as few terms as possible)

Summary

- Can use regression to predict un-measured values
- Before fit
  - Visual relationship (scatter plot) and residual analysis
- Strength of fit – Coefficient of Determination (R^2) and correlation (R)
- Beware
  - Correlation is not causation
  - Extrapolation
- Higher order, more complex models can fit better
  - Beware of overfit → less predictive power