Homework #7
Solutions

#1. Use the pumping lemma for CFL's to show $L = \{a^i b^j a^i b^j \mid i, j \geq 0\}$ is not a CFL.

Proof by contradiction using the Pumping Lemma.

Assume L is context-free. L is obviously infinite, since i, j > 0. Pick a string *w* in *L*

with length |w| > m, for example, $w = a^m b^m a^m b^m$, j > 0.

Then w = uvxyz with

$|vxy| < m$ and $|vy| > 1$ and uv^ixy^iz is also in L for all $i \geq 0$

1) vxy is within first a^m
$uv^2xy^2z = a^{m+\|v\|+\|y\|}b^ma^mb^m \notin L$ because $m+\|v\|+\|y\| \neq m$ because $\|vy\| > 0$
2) vxy is within first b^m
$uv^2xy^2z = a^mb^{m+\|v\|+\|y\|}a^mb^m \notin L$ … similar to #1
3) vxy overlaps first a^m and first b^m
3a) v itself straddles a's and b's: pumping v would produce strings of the form $a…a(ab)^ib…ba^mb^m$ which are not in L
3b) y itself straddles a's and b's: similar to #3a
3c) v is all a's, y is all b's: $uv^2xy^2z = a^{m+\|v\|}b^{m+\|y\|}a^mb^m \notin L$ because either $m+\|v\| \neq m$, or $m+\|y\| \neq m$
4) vxy overlaps first b^m and second a^m: Similar logic to #3
5) vxy is within 2^{nd} a^m: Same as #1
6) vxy is within 2^{nd} b^m: Same as #2
7) vxy overlaps 2^{nd} a^m and 2^{nd} b^m: Same as #3

#2. Consider the following 2 languages:

$L_1 = \{a^nb^{2n}c^m \mid n, m \geq 0\}$
$L_2 = \{a^nb^mc^{2m} \mid n, m > \underline{0}\}$

a) Show that each of these languages is context-free.

G_1
S→AC| ε

G_2
S→AB| ε

$A \to aAbb \mid \varepsilon$ **$A \to Aa \mid \varepsilon$**
$C \to cC \mid \varepsilon$ **$B \to bBcc \mid \varepsilon$**

L_1: |b's| = 2×|a's|
L_2: |c's| = 2×|b's|

So $L_1 \cap L_2 = \{ a^n b^{2n} c^{4n} \mid n \geq 0 \}$

b) Is $L_1 \cap L_2$ context-free? Justify your answer.

Not context-free (easy use of pumping lemma.) Pick w = $a^m b^{2m} c^{4m}$

#3. Convert the following grammar to Chomsky Normal Form

$S \to A \mid A\,B\,a \mid A\,b\,A$
$A \to A\,a \mid \varepsilon$
$B \to B\,b \mid BC$
$C \to C\,B \mid C\,A \mid b\,B$

1. Eliminate ε-productions:

$S \to A \mid A\,B\,a \mid A\,b\,A \mid B\,a \mid b\,A \mid A\,b \mid \varepsilon$
$A \to A\,a \mid a$
$B \to B\,b \mid BC$
$C \to C\,B \mid C\,A \mid b\,B \mid C$

2. Remove unit productions $S \to A\,a \mid a \mid A\,B\,a \mid A\,b\,A \mid B\,a \mid b\,A \mid A\,b \mid \varepsilon$
$A \to A\,a \mid a$
$B \to B\,b \mid BC$
$C \to C\,B \mid C\,A \mid b\,B$

3. a) Eliminate useless (non-generating) symbols and all productions involving one or more of those symbols. $S \to A\,a \mid a \mid A\,b\,A \mid b\,A \mid A\,b$
$A \to A\,a \mid a$
b) Eliminate unreachable symbols: None

4. Convert to CNF:

$S \to \varepsilon \mid A\,A_1 \mid A\,B \mid B_1\,A \mid A\,B_1$
$A_1 \to a$
$B \to B_1 A$

$B_1 \to b$
$A \to A\,A_1 \mid a$

#4. Let G be a grammar in Chomsky Normal Form. Fill in the following table.

w	|w|	length(derivation)	max depth (tree)	min depth(tree)
ε	0	1	1	1
a_1	1	1	1	1
a_1a_2	2	3	2	2
$a_1a_2a_3$	3	5	3	3
$a_1a_2a_3a_4$	4	7	4	3
$a_1a_2a_3a_4a_5$	5	8	5	4